# A Stroll through Partial Fields

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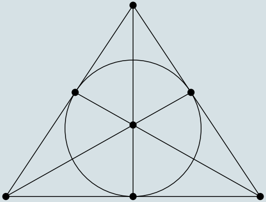
# Matroid representation

## Matroid



- Finite set of “elements”
- Every subset is either dependent or independent

## Representation

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

- Set of vectors (matrix columns)
- Linear dependence

Example is representable over $\mathbb{F}$ if and only if $\chi(\mathbb{F}) = 2$.

# Matroid representation

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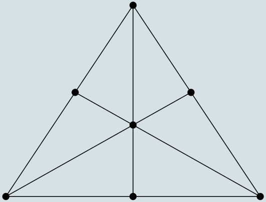

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Example is representable over $\mathbb{F}$ if and only if $\chi(\mathbb{F}) \neq 2$.

# Sets of fields

Today: given a representation over some fields, are there more?

Prototypical result:

### Theorem (Tutte 1965)

*Let $M$ be a matroid. The following are equivalent:*

- *$M$ is representable over both* GF(2) *and* GF(3)
- *$M$ is representable over $\mathbb{R}$ by a totally unimodular matrix*
- *$M$ is representable over every field*

# Totally unimodular matrices

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

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$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \text{-1} & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Every square submatrix has determinant in $\{0, 1, -1\}$.

# Determinants of submatrices

Note: if there is a representation of a matroid, then there is one of the form $[I|D]$.

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$\Rightarrow$ If $[I|D_1], [I|D_2]$ represent the same matroid, then $D_1, D_2$ have same *determinant structure*.

## Theorem (Tutte 1965)

*Let $M$ be a matroid. The following are equivalent:*

1. *$M$ is representable over both* GF(2) *and* GF(3)
2. *$M$ is representable over $\mathbb{R}$ by a totally unimodular matrix*
3. *$M$ is representable over every field*

Main step: (1) $\Rightarrow$ (2).

# Ternary matroids

- $\sqrt[6]{1}$: every determinant in $\{0\} \cup \{x \in \mathbb{C} \mid x^6 = 1\}$.
- *Dyadic:* every determinant in $\{0\} \cup \{\pm 2^i \mid i \in \mathbb{Z}\}$.
- *Near-regular*: every determinant in $\{0\} \cup \{\pm \alpha^i (\alpha - 1)^j \mid i, j \in \mathbb{Z}\}$.

## Theorem (Whittle 1994, 1997)

- GF(3) × GF(4)-*representable* ⇔ $\sqrt[6]{1}$.
- GF(3) × GF(5)-*representable* ⇔ *dyadic* ⇔ *representable over* GF($p$) *for all odd primes.*
- GF(3) × GF(4) × GF(5)-*representable* ⇔ *near-regular* ⇔ *representable over all fields with* ≥ 3 *elements.*

# Golden Mean matroids

Consider

$$\{0\} \cup \{\pm r^i \mid i \in \mathbb{Z}\}$$

where $r = \frac{1}{2}(1+\sqrt{5})$, i.e. a root of $r^2 - r - 1$. These are the units of the *ring of integers* $\mathbb{O}$ of $\mathbb{Q}(\sqrt{5})$.

## Theorem (Vertigan (unpublished); vZ,Pendavingh 2007)

*Let $M$ be a matroid. The following are equivalent:*

1. *$M$ is representable over both* GF(4) *and* GF(5)
2. *$M$ is representable over $\mathbb{R}$ by a Golden Mean matrix.*
3. *$M$ is representable over all* GF($p$) *where $p$ is* 0 *or a quadratic residue* $\bmod 5$.

Main difficulty: (1) $\Rightarrow$ (2).

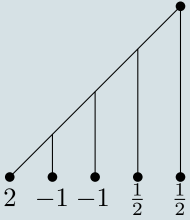
# Partial field axioms (Semple and Whittle 1996)

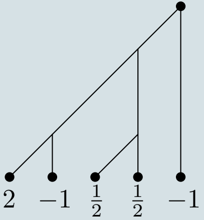
Partial field: structure $(\mathbb{P}, +, \cdot, 0, 1)$ satisfying

1. $\mathbb{P} - \{0\}$ is an abelian group under $\cdot$.
2. For all $p$, $p + 0 = p$.
3. For all $p$, there is a $(-p)$ such that $p + (-p) = 0$.
4. For all $p, q$, if $p + q$ defined, then $q + p$ defined and $p + q = q + p$.
5. For all $p, q, r$: $p(q + r)$ defined $\Leftrightarrow$ $pq + pr$ is defined. Then $p(q + r) = pq + pr$.
6. The *associative law* holds for $+$.

# The associative law

Want: sums of more than 2 elements (determinant computations!)



# $\mathbb{P}$-matrices

$A$ is a $\mathbb{P}$-*matrix* if it is a matrix over $\mathbb{P}$ such that, for every $k$, for every $k \times k$ submatrix $C$,

$$\det(C) := \sum_{\sigma} \operatorname{sgn}(\sigma) c_{1\sigma(1)} c_{2\sigma(2)} \cdots c_{k\sigma(k)}$$

is defined.

# Sources of partial fields

- Let $\mathbb{G}$ be the units of a commutative ring $\mathbb{O}$. Then
$$(\mathbb{G} \cup \{0\}, +, \cdot, 0, 1)$$
is a partial field. Notation: $\mathbb{P}(\mathbb{O})$.
- Let $\mathbb{G}$ be a multiplicative subgroup of a partial field, with $-1 \in \mathbb{G}$. Then
$$(\mathbb{G} \cup \{0\}, +, \cdot, 0, 1)$$
is a partial field.

# Examples of partial fields

- Regular, dyadic, $\sqrt[6]{1}$, near-regular, golden mean
- $\mathbb{P}(\mathbb{F}_1 \times \cdots \times \mathbb{F}_k)$

## Lemma

*A matroid M is representable over all of $\mathbb{F}_1, \ldots, \mathbb{F}_k$ if and only if it is representable over*

$$\mathbb{P}(\mathbb{F}_1 \times \cdots \times \mathbb{F}_k).$$

$\Rightarrow$ proving our theorems reduces to comparing partial fields.

# Partial field homomorphisms

## Definition

$\varphi : \mathbb{P}_1 \to \mathbb{P}_2$ is a partial field homomorphism if

- $\varphi(pq) = \varphi(p)\varphi(q)$
- If $p + q$ is defined, then $\varphi(p) + \varphi(q)$ is defined and equal to $\varphi(p+q)$.

$\varphi$ is nontrivial if $\varphi(1) \neq 0$.

## Theorem (Semple and Whittle, 1996)

*If $A$ is a $\mathbb{P}_1$-matrix and $\varphi$ is nontrivial, then $\varphi(A)$ is a $\mathbb{P}_2$-matrix; for a submatrix $C$ we have*

$$\det(C) = 0 \Leftrightarrow \det(\varphi(C)) = 0.$$

# Homomorphisms: applications

## Theorem (Summary)

- $\mathbb{P}(\mathrm{GF}(2) \times \mathrm{GF}(3))$-*representable* $\Leftrightarrow$ *regular.*
- $\mathbb{P}(\mathrm{GF}(3) \times \mathrm{GF}(4))$-*representable* $\Leftrightarrow$ $\sqrt[6]{1}$.
- $\mathbb{P}(\mathrm{GF}(3) \times \mathrm{GF}(5))$-*representable* $\Leftrightarrow$ *dyadic.*
- $\mathbb{P}(\mathrm{GF}(3) \times \mathrm{GF}(4) \times \mathrm{GF}(5))$-*representable* $\Leftrightarrow$ *near-regular.*
- $\mathbb{P}(\mathrm{GF}(4) \times \mathrm{GF}(5))$-*representable* $\Leftrightarrow$ *golden mean.*

## Proof.

- Use homomorphism for $\Leftarrow$ of all five, and $\Rightarrow$ of first two.
- Other implications: lot of (matroid-theoretic) work.

□

# What partial fields really are

Theorem (Vertigan (unpublished); vZ, 2007)

*Every partial field can be obtained from a multiplicative subgroup of the units of some ring.*

Note: partial field homomorphisms do not extend to ring homomorphisms!

# Where to go from here?

## Conjecture

*A matroid M is representable over any field with* $\geq k$ *elements* $\Leftrightarrow$ *it is representable over all fields* $\mathrm{GF}(q)$ *with* $k \leq q \leq n_k$.

Generalizes "regular" and "near-regular". Need: the right "big" partial field.

## Question

*Can every (interesting) partial field be obtained from the units of an integral domain?*

## Quest

*Find other nice partial fields!*

# The end

Thank you.