A Stroll through Partial Fields

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Matroid representation

Matroid

- Finite set of “elements”
- Every subset is either dependent or independent

Representation

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}$$

- Set of vectors (matrix columns)
- Linear dependence

Example is representable over $\mathbb{F}$ if and only if $\chi(\mathbb{F}) = 2$. 
Matroid representation

Matroid

- Finite set of “elements”
- Every subset is either dependent or independent

Representation

Set of vectors (matrix columns)

Linear dependence

Example is representable over $\mathbb{F}$ if and only if $\chi(\mathbb{F}) \neq 2$. 
Sets of fields

Today: given a representation over some fields, are there more?

Prototypical result:

**Theorem (Tutte 1965)**

Let $M$ be a matroid. The following are equivalent:

- $M$ is representable over both $GF(2)$ and $GF(3)$
- $M$ is representable over $\mathbb{R}$ by a totally unimodular matrix
- $M$ is representable over every field
Totally unimodular matrices

\[
\begin{pmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1 \\
\end{pmatrix}
\]

Every square submatrix has determinant in \( \{0 \)
Totally unimodular matrices

\[
\begin{bmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1
\end{bmatrix}
\]

Every square submatrix has determinant in \{0, 1\}
Totally unimodular matrices

$$\begin{pmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1
\end{pmatrix}$$

Every square submatrix has determinant in \{0, 1, -1\}.
Determinants of submatrices

Note: if there is a representation of a matroid, then there is one of the form $[I|D]$.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}
$$

⇒ If $[I|D_1]$, $[I|D_2]$ represent the same matroid, then $D_1, D_2$ have same determinant structure.
Theorem (Tutte 1965)

Let $M$ be a matroid. The following are equivalent:

1. $M$ is representable over both $\text{GF}(2)$ and $\text{GF}(3)$
2. $M$ is representable over $\mathbb{R}$ by a totally unimodular matrix
3. $M$ is representable over every field

Main step: $(1) \Rightarrow (2)$. 
Ternary matroids

- $^6\sqrt{1}$: every determinant in $\{0\} \cup \{x \in \mathbb{C} \mid x^6 = 1\}$.
- Dyadic: every determinant in $\{0\} \cup \{\pm 2^i \mid i \in \mathbb{Z}\}$.
- Near-regular: every determinant in $\{0\} \cup \{\pm \alpha^i(\alpha - 1)^j \mid i, j \in \mathbb{Z}\}$.

Theorem (Whittle 1994, 1997)

- $\text{GF}(3) \times \text{GF}(4)$-representable $\iff ^6\sqrt{1}$.
- $\text{GF}(3) \times \text{GF}(5)$-representable $\iff \text{dyadic} \iff \text{representable over } \text{GF}(p) \text{ for all odd primes}$.
- $\text{GF}(3) \times \text{GF}(4) \times \text{GF}(5)$-representable $\iff \text{near-regular} \iff \text{representable over all fields with } \geq 3 \text{ elements.}$
Golden Mean matroids

Consider

\[ \{0\} \cup \{\pm r^i \mid i \in \mathbb{Z}\} \]

where \( r = \frac{1}{2}(1 + \sqrt{5}) \), i.e. a root of \( r^2 - r - 1 \). These are the units of the ring of integers \( \mathcal{O} \) of \( \mathbb{Q}(\sqrt{5}) \).

Theorem (Vertigan (unpublished); vZ, Pendavingh 2007)

Let \( M \) be a matroid. The following are equivalent:

1. \( M \) is representable over both GF(4) and GF(5)
2. \( M \) is representable over \( \mathbb{R} \) by a Golden Mean matrix.
3. \( M \) is representable over all GF(p) where \( p \) is 0 or a quadratic residue \( \mod 5 \).

Main difficulty: \( (1) \Rightarrow (2) \).
Partial field axioms (Semple and Whittle 1996)

Partial field: structure \((\mathbb{P}, +, \cdot, 0, 1)\) satisfying

1. \(\mathbb{P} - \{0\}\) is an abelian group under \(\cdot\).
2. For all \(p\), \(p + 0 = p\).
3. For all \(p\), there is a \((-p)\) such that \(p + (-p) = 0\).
4. For all \(p, q\), if \(p + q\) defined, then \(q + p\) defined and
\(p + q = q + p\).
5. For all \(p, q, r\): \(p(q + r)\) defined \(\iff\) \(pq + pr\) is defined. Then
\(p(q + r) = pq + pr\).
6. The associative law holds for +.
The associative law

Want: sums of more than 2 elements (determinant computations!)

\[
\begin{bmatrix}
2 & -1 & -1 & \frac{1}{2} & \frac{1}{2} \\
\end{bmatrix}
\quad \begin{bmatrix}
2 & -1 & \frac{1}{2} & \frac{1}{2} & -1 \\
\end{bmatrix}
\]
A is a $\mathbb{P}$-matrix if it is a matrix over $\mathbb{P}$ such that, for every $k$, for every $k \times k$ submatrix $C$,

$$\det(C) := \sum_{\sigma} \text{sgn}(\sigma)c_{1\sigma(1)}c_{2\sigma(2)} \cdots c_{k\sigma(k)}$$

is defined.
Sources of partial fields

- Let $G$ be the units of a commutative ring $\mathcal{O}$. Then
  \[(G \cup \{0\}, +, \cdot, 0, 1)\]
is a partial field. Notation: $\mathbb{P}(\mathcal{O})$.

- Let $G$ be a multiplicative subgroup of a partial field, with $-1 \in G$. Then
  \[(G \cup \{0\}, +, \cdot, 0, 1)\]
is a partial field.
Examples of partial fields

- Regular, dyadic, $\sqrt[6]{1}$, near-regular, golden mean
- $\mathbb{P}(F_1 \times \cdots \times F_k)$

Lemma

A matroid $M$ is representable over all of $F_1, \ldots, F_k$ if and only if it is representable over

$$\mathbb{P}(F_1 \times \cdots \times F_k).$$

$\Rightarrow$ proving our theorems reduces to comparing partial fields.
Partial field homomorphisms

Definition
\( \varphi : P_1 \rightarrow P_2 \) is a partial field homomorphism if
- \( \varphi(pq) = \varphi(p)\varphi(q) \)
- If \( p + q \) is defined, then \( \varphi(p) + \varphi(q) \) is defined and equal to \( \varphi(p + q) \).

\( \varphi \) is nontrivial if \( \varphi(1) \neq 0 \).

Theorem (Semple and Whittle, 1996)
If \( A \) is a \( P_1 \)-matrix and \( \varphi \) is nontrivial, then \( \varphi(A) \) is a \( P_2 \)-matrix; for a submatrix \( C \) we have

\[
\det(C) = 0 \iff \det(\varphi(C)) = 0.
\]
Homomorphisms: applications

Theorem (Summary)

- $\mathbb{P}(\text{GF}(2) \times \text{GF}(3))$-representable $\iff$ regular.
- $\mathbb{P}(\text{GF}(3) \times \text{GF}(4))$-representable $\iff 6\sqrt{I}$.
- $\mathbb{P}(\text{GF}(3) \times \text{GF}(5))$-representable $\iff$ dyadic.
- $\mathbb{P}(\text{GF}(3) \times \text{GF}(4) \times \text{GF}(5))$-representable $\iff$ near-regular.
- $\mathbb{P}(\text{GF}(4) \times \text{GF}(5))$-representable $\iff$ golden mean.

Proof.

- Use homomorphism for $\iff$ of all five, and $\Rightarrow$ of first two.
- Other implications: lot of (matroid-theoretic) work.
What partial fields really are

Theorem (Vertigan (unpublished); vZ, 2007)

*Every partial field can be obtained from a multiplicative subgroup of the units of some ring.*

Note: partial field homomorphisms do not extend to ring homomorphisms!
Where to go from here?

Conjecture
A matroid $M$ is representable over any field with $\geq k$ elements $\iff$ it is representable over all fields $\text{GF}(q)$ with $k \leq q \leq n_k$.

Generalizes “regular” and “near-regular”. Need: the right “big” partial field.

Question
Can every (interesting) partial field be obtained from the units of an integral domain?

Quest
Find other nice partial fields!
Thank you.