Where innovation starts

Universal Representations of Matroids

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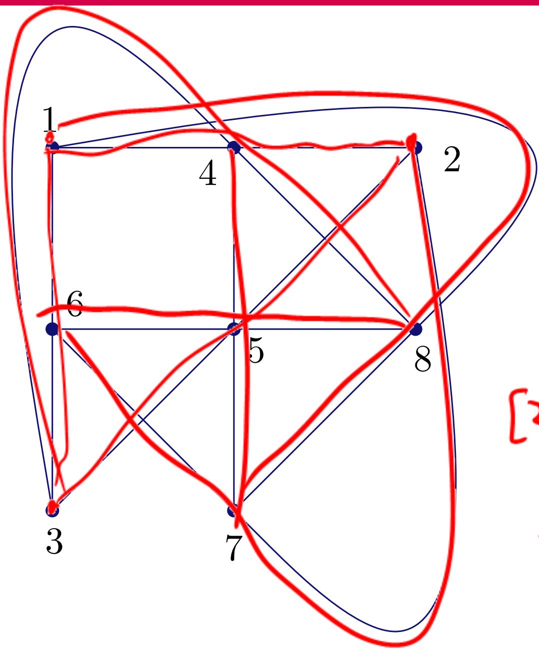
Example



dep. sets

* 2 incident pts.
* 3 on a line
* 4 in a plane

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & u \\ 1 & 0 & v \end{vmatrix} = u - 1 = 0 \quad (3\ 4\ 8)$$

[267]: $z - x = 0$

$x - v + 1 = 0 \qquad v = x + 1$

$1 - y + x = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & x+1 & 1 \\ 0 & x & x+1 \end{vmatrix} \quad (1\ 7\ 8)$$

$= (x+1)^2 - x$

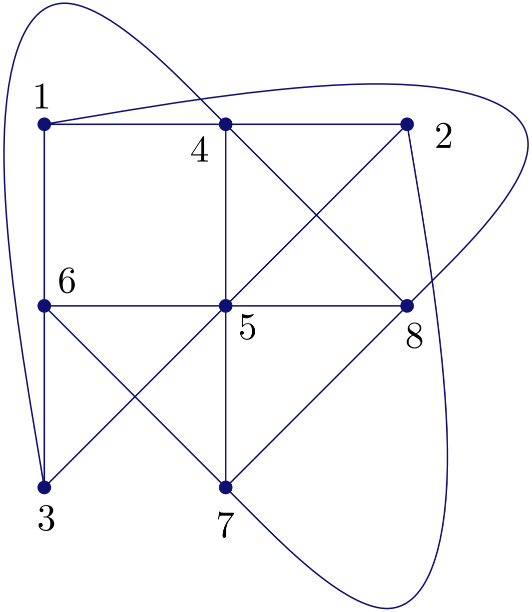
$= x^2 + x + 1$

$= 0$

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & x+1 & 1 \\ 0 & 0 & 1 & 0 & 1 & x & x & x+1 \end{bmatrix}$$

Example



$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & x+1 & 1 \\ 0 & 0 & 1 & 0 & 1 & x & x & x+1 \end{bmatrix}$$

The topic for today

- How to find (all) representations of “combinatorial geometries”.

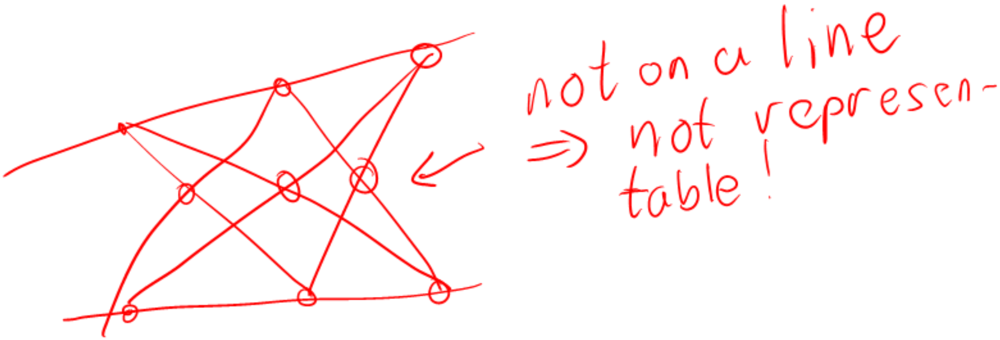
What is a matroid?

A matroid is a pair $(E, \mathcal{B})$, where E is a finite set and $\mathcal{B}$ a collection of subsets of E, the *bases*, satisfying the axioms

(*i*) $\mathcal{B}$ is nonempty;

(*ii*) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \setminus B_2$, then there exists a $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

Example: E is a set of vectors in some vector space. Note that not all matroids arise in this way!



Representability

If A is $r \times E$ matrix over field $\mathbb{F}$, then $M(A) = (E, \mathcal{B})$ is matroid with $E = \{$columns of $A\}$ and $\mathcal{B} = \{$maximal linearly independent sets$\}$. $M(A)$ invariant under

(i) Swapping columns *and labels*

(ii) Row operations

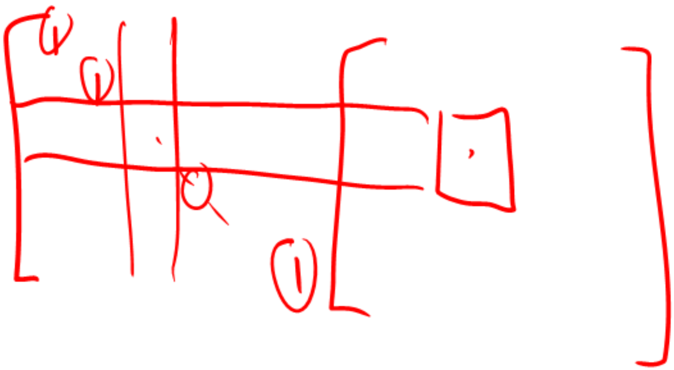
(iii) Column scaling

Standard representation: some basis forms identity matrix.

$$\begin{array}{c} \quad X \qquad\qquad Y \\ \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & A' \\ 0 & & 1 & \end{array}\right] \end{array} \longleftrightarrow X \begin{array}{c} Y \\ \left[\begin{array}{c} \\ A' \\ \\ \end{array}\right] \end{array}$$

Standard representation

Note: 1-1 correspondence between square submatrices of A and $r \times r$ submatrices of $[I|A]$.
Notation: suppose $X' \subseteq X, Y' \subseteq Y$. Then $A[X', Y']$ is restriction of A to rows X', columns Y'. If $Z \subseteq X \cup Y$ then $A[Z] = A[X \setminus Z, Y \cap Z]$.
Now a set $Z \subseteq X \cup Y$ is a basis of $M([I|A])$ if and only if $|Z| = r$ and $\det(A[Z]) \neq 0$.



Pivots

Move between standard representations by *pivoting* on a nonzero entry:

$$A = \begin{array}{c} \\ x \\ X' \end{array} \begin{array}{c} \begin{array}{cc} y & Y' \end{array} \\ \left[\begin{array}{c|c} a & b \\ \hline c & D \end{array}\right] \end{array} \to \begin{array}{c} \\ y \\ X' \end{array} \begin{array}{c} \begin{array}{cc} x & Y' \end{array} \\ \left[\begin{array}{c|c} a^{-1} & a^{-1}b \\ \hline -a^{-1}c & D - a^{-1}cb \end{array}\right] \end{array} = A^{xy}.$$

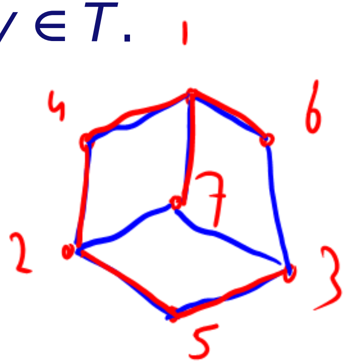
In matrix $[I|A]$ this is row reduction followed by column exchange.

Normalization

Choose basis X. A is an $X \times Y$ rep. matrix over $\mathbb{F}$.

- Positions of zeroes are fixed.
- $G(A)$ is bipartite graph with vertex classes X, Y. $xy \in E(G)$ if and only if $A_{xy} \neq 0$.
- Suppose edges $x_1y_1, \ldots, x_ky_k$ form spanning forest of $G(A)$. Let $\theta_1, \ldots, \theta_k \in \mathbb{F}^*$. Can scale rows and columns so that $A_{x_iy_i} = \theta_i$ for all i.
- A is *normalized* if, for some spanning forest T, $A_{xy} = 1$ for all $xy \in T$.

	4	5	6	7
1	1	0	1	1
2	1	1	0	1
3	0	1	-1	1



Universal representation

$M = (X \cup Y, \mathcal{B})$; X basis. $A_{M,X} = (a_{ij})$ $X \times Y$ matrix; a_{ij} unknowns. For each $B \in \mathcal{B}$ an unknown i_B. Ring $\mathbb{Z}[\{a_{ij}\} \cup \{i_B\}]$. Construct ideal $\mathcal{I}$:

- If $(X \setminus i) \cup j \notin \mathcal{B}$ then $a_{ij} \in \mathcal{I}$;
- T spanning forest of $G(A)$. If $ij \in T$ then $a_{ij} - 1 \in \mathcal{I}$;
- $Z \subseteq X \cup Y$, $|Z| = r$. If $Z \notin \mathcal{B}$ then $\det(A_{M,X}[Z]) \in \mathcal{I}$;
- $Z \subseteq X \cup Y$, $|Z| = r$. If $Z \in \mathcal{B}$ then $\det(A_{M,X}[Z]) i_Z - 1 \in \mathcal{I}$.

$$\overline{\mathbb{B}}_M := \mathbb{Z}[\{a_{ij}\} \cup \{i_B\}]/\mathcal{I}.$$

(Fenton 1984, mostly)

Universal representation

Theorem 1. *If A is an $X \times Y$ matrix over field $\mathbb{F}$ such that $M = M([I|A])$, and A is T-normalized, then there is a ring homomorphism $\varphi : \overline{\mathbb{B}}_M \to \mathbb{F}$ such that*

$$\varphi(A_{M,X}) = A.$$

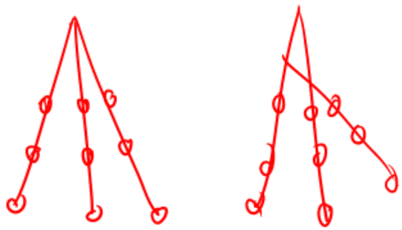
Theorem 2. *$\overline{\mathbb{B}}_M$ does not depend on choice of X or T (different choices give isomorphic rings).*

Some open problems

While this construction works for any fixed matroid, often *classes* of matroids are of interest. For example:

- Free spikes, free swirls (there's one of each for each rank r);
- One-element co-extensions of PG$(2, q)$, that are still representable over GF(q);
- Uniform matroids (cf. "The Main Conjecture for MDS codes").

$U_{r,n}$



Classification problems

- If M_1, M_2 are matroids, how to test if $\overline{\mathbb{B}}_{M_1} \cong \overline{\mathbb{B}}_{M_2}$?
- What rings can occur as $\overline{\mathbb{B}}_M$?

Theorem 3 (Tutte 1965)**.** *If M is binary and 3-connected, then $\overline{\mathbb{B}}_M \cong \mathrm{GF}(2)$ or $\overline{\mathbb{B}}_M \cong \mathbb{Z}$.*

Theorem 4 (Whittle 1997)**.** *If M is ternary, nonbinary, and 3-connected, then $\overline{\mathbb{B}}_M \cong \mathrm{GF}(3)$ or $\mathbb{Z}[1/2]$ or $\mathbb{Z}[\zeta]$ or $\mathbb{Z}[\alpha, 1/\alpha, 1-\alpha, 1/(1-\alpha)]$. Here ζ is a root of $x^2 - x + 1$.*

Note: no finite list for $\mathrm{GF}(q)$, $q \geq 4$. Otherwise no results known for nonbinary and nonternary classes.

Minors

A' is a *minor* of A (notation: $A' \preceq A$) if A' can be obtained from A by a sequence of the following operations:

(*i*) Multiplying the entries of a row or column by a unit;

(*ii*) Deleting rows or columns;

(*iii*) Permuting rows or columns (together with labels);

(*iv*) Pivoting over a nonzero entry.

Cross ratios

$$\mathrm{Cr}(A) := \left\{ p : \begin{bmatrix} 1 & 1 \\ p & 1 \end{bmatrix} \preceq A \right\}.$$

Occur in pairs of (at most) six:

$$\left\{ p, 1-p, \frac{1}{1-p}, \frac{p}{p-1}, \frac{p-1}{p}, \frac{1}{p} \right\}.$$

Theorem 5. $\overline{\mathbb{B}}_M$ *equals the subring generated by* $\mathrm{Cr}(A_{M,X})$*.*

Lifts

Fundamental elts.: $\mathcal{F}(\mathbb{B}_M) := \{p \in \mathbb{B}_M^* \mid 1 - p \in \mathbb{B}_M^*\}$.
$\widetilde{F} := \{\widetilde{p} \mid p \in \mathcal{F}(\mathbb{B}_M)\}$ set of symbols, I ideal in $\mathbb{Z}[\widetilde{F}]$ generated by

(*i*) $\widetilde{0} - 0$; $\widetilde{1} - 1$;

(*ii*) $\widetilde{-1} + 1$ if $-1 \in \mathcal{F}(\mathbb{B}_M)$;

(*iii*) $\widetilde{p} + \widetilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{B}_M)$, $p + q = 1$;

(*iv*) $\widetilde{p}\widetilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{B}_M)$, $pq = 1$;

(*v*) $\widetilde{p}\widetilde{q}\widetilde{r} - 1$, where $p, q, r \in \mathrm{Cr}(A_{M,X})$, $pqr = 1$, and

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q^{-1} \end{bmatrix} \preceq A_{M,X}. \tag{1}$$

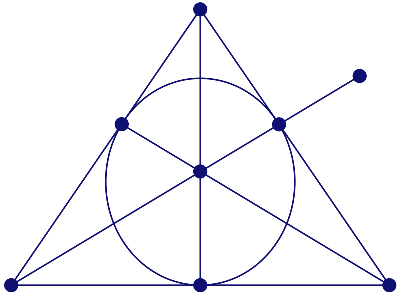
Theorem 6. *M is representable over $\mathbb{Z}[\widetilde{F}]/I$.*

Obstacles

For this to be useful beyond ternary matroids, need:

- Understand the set $\mathcal{F}(\mathbb{B}_M)$, or
- Manage to replace $\mathcal{F}(\mathbb{P})$ by Cr(A) throughout and characterize relations (*iii*), (*iv*) in terms of minors.

Nasty example: universal partial field of following configuration is GF(2)[α, ~~$1-\alpha$~~, $1/\alpha$, $1/(1-\alpha)$]. The set of fundamental elements is *infinite*, since $1-\alpha^{2^k} = (1-\alpha)^{2^k}$.



The start.

Suggestions?