New Korkin–Zolotarev Inequalities

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1. Sphere Packings and Quadratic Forms

The geometry of numbers is a field of mathematics initiated and named by Minkowski. The main objects studied are *lattices*, discrete subgroups of $\mathbb{R}^n$. A classical problem is the search for a lattice with a dense *sphere packing*. Hermite's constant γ_n is a measure for the maximum density of a lattice sphere packing in dimension n. This constant has been determined exactly for $n \le 8$ and $n = 24$. All relevant information is captured by the *quadratic form* associated with the lattice. It has the following, unique, Lagrange expansion:

$$q(x_1,\ldots,x_n) = \sum_{i=1}^{n} A_i \Big(x_i - \sum_{j>i} \alpha_{ij} x_j \Big)^2. \qquad (1)$$

Quadratic forms q, q' are equivalent if $q'(x) = q(Ux)$ for some unimodular matrix U. A form is *Korkin–Zolotarev reduced* if

$$|\alpha_{ij}| \le \frac{1}{2} \text{ for all } i,j, \text{ and } \alpha_{i,i+1} \ge 0 \text{ for all } i; \text{ and} \qquad (\mathbf{S})$$

$$A_k \le \sum_{i=k}^{n} A_i \Big(x_i - \sum_{j>i} \alpha_{ij} x_j \Big)^2 \text{ for all nonzero } x \in \mathbb{Z}^{n-k+1}, k = 1,\ldots,n-1. \qquad (\mathbf{M})$$

Hermite's constant, in terms of quadratic forms, is defined as

$$\gamma_n := \max \left\{ \frac{\min\{q(x) \mid x \in \mathbb{Z}^n, x \neq 0\}}{\det(q)^{1/n}} \mid q \text{ is an } n\text{-ary positive definite quadratic form} \right\}. \qquad (2)$$

Each quadratic form is equivalent to a KZ-reduced one. Therefore we have

$$\gamma_n = \max \left\{ \frac{A_1}{(\prod_i A_i)^{1/n}} \mid (A_1,\ldots,A_n) \text{ are outer coefficients for some KZ-reduced form } q \right\}. \qquad (3)$$

Like Korkin and Zolotarev [1], we will study the feasible set of this maximization problem to obtain upper bounds.

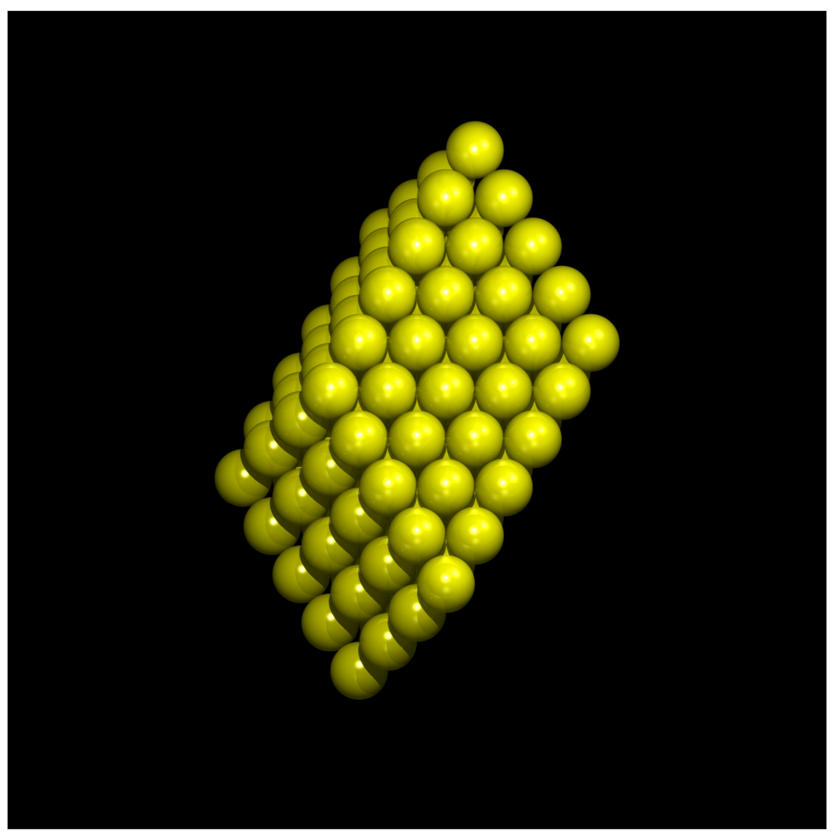

Figure 1: *Part of a lattice sphere packing in 3 dimensions*

2. Semidefinite Programming

Important observation: only finitely many inequalities from (**M**) are sufficient to characterize the KZ-reduced forms:

Theorem 1 (Novikova [3]). *For each $n \ge 2$, there is a finite set $X_n \subseteq \mathbb{Z}^n$ such that an n-ary form with Lagrange expansion (1) is KZ-reduced if and only if $\sum_{i=2}^{n} A_i \big(x_i - \sum_{j>i} \alpha_{ij} x_j \big)^2$ is KZ-reduced,* (**S**) *holds and*

$$A_1 \le q(x) \text{ for all } x \in X_n. \qquad (4)$$

We want to find linear inequalities bounding the feasible set of (3). This leads to linear optimization problems on the semialgebraic set defined by (**S**) and the finite subset of (**M**). We construct a semidefinite programming relaxation of this problem, following [2]. We improve the lower bound by *branch and bound*:

- Pick i, j.
- Split the domain of $\alpha_{i,j}$, defined in (**S**), in two parts.
- Solve the two resulting optimization problems.

Each problem yields a bound on the optimum over that smaller set; the worst will bound the original problem. Repeating this we get very good approximations to the optimum. Efficiency depends on how i and j are picked.
Each bound can be certified by a solution to the dual of that SDP. The dual solution can be approximated by a rational vector. This was used to prove Theorem 2 rigorously.

References

[1] A. Korkine and G. Zolotareff, *Sur les formes quadratiques*, Math. Ann., 6 (1873), pp. 366–389.

[2] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim., 11 (2001), pp. 796–817.

[3] N. V. Novikova, *Domains of Korkin-Zolotarev reduction of positive quadratic forms in $n \le 8$ variables and reduction algorithms for these domains*, Dokl. Akad. Nauk SSSR, 270 (1983), pp. 48–51.

[4] R. A. Pendavingh and S. H. M. van Zwam, *New Korkin–Zolotarev inequalities*, SIAM J. Optim., 18 (2007), pp. 364–378. Results and software at `http://www.win.tue.nl/kz/`.

3. Results

Korkin and Zolotarev [1] proved

$$A_{i+1} \ge \frac{3}{4} A_i, \text{ and } A_{i+2} \ge \frac{2}{3} A_i. \qquad (5)$$

Hermite's constant can be bounded by

$$\gamma_n^n \le \max \left\{ \frac{A_1^n}{\prod_{i=1}^{n} A_i} \mid (5), A_1 = 1 \right\}, \qquad (6)$$

which is exact for $n \le 4$. The maximum is necessarily attained at a vertex of the polyhedron. For larger n, bounds were obtained by other methods. We proved the following new inequalities:

Theorem 2. *If $(A_1,\ldots,A_n)$ are the outer coefficients of a KZ-reduced quadratic form, and $n \ge 4$, then*

$$\begin{aligned}
-25A_1 - 36A_2 + 48A_3 + 40A_4 &\ge -7 \cdot 10^{-6} A_4 && (7)\\
-5A_1 + 2A_4 + 8A_5 &\ge -3 \cdot 10^{-4} A_5 && (8)\\
-4A_1 - 3A_3 + 4A_4 + 8A_5 &\ge -5 \cdot 10^{-5} A_5 && (9)
\end{aligned}$$

Conjecture 1. *All right-hand sides above can be improved to 0.*

If this is true, these inequalities give the exact bound on Hermite's constant for $n \le 8$ using the analog of (6). Main open problem: find, and prove, suitable new inequalities for $n = 9$ or 10.

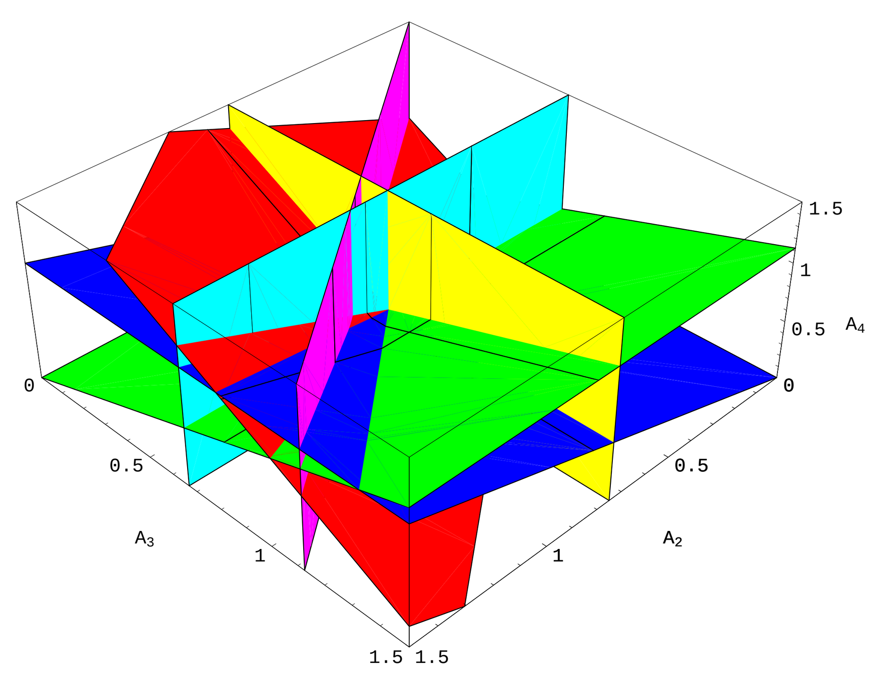

Figure 2: *Inequalities on the outer coefficients for $n = 4$, where $A_1 = 1$. The red plane corresponds to (7).*