Confinement to sub-partial fields

Stefan van Zwam
Based on joint work with Rudi Pendavingh

Technische Universiteit Eindhoven
University of Technology

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Minors of matrices

Let A be an $X \times Y$ matrix. A *minor* of A is a matrix obtained by

- Scaling rows and columns;
- Deleting rows and columns (notation: $A - x$);
- Pivoting on a nonzero entry:

$$A = \begin{array}{c} \\ x \\ X' \end{array} \begin{array}{c} \begin{array}{cc} y & Y' \end{array} \\ \left[\begin{array}{c|c} a & b \\ \hline c & D \end{array}\right] \end{array} \to \begin{array}{c} \\ y \\ X' \end{array} \begin{array}{c} \begin{array}{cc} x & Y' \end{array} \\ \left[\begin{array}{c|c} a^{-1} & a^{-1}b \\ \hline -a^{-1}c & D - a^{-1}cb \end{array}\right] \end{array} = A^{xy}$$

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Correspond with minors of matroid $M([I|A])$.

$$\begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & A \\ 0 & & 1 & \end{array}\right] \end{array} \longleftrightarrow X \begin{array}{c} Y \\ \left[\begin{array}{c} A \end{array}\right] \end{array}$$

Previously...

- A partial field $\mathbb{P}$ is a pair $(\mathbb{O}, \mathbf{G})$ of a ring and a group, such that

$$-1 \in \mathbf{G} \subseteq \mathbb{O}^*.$$

  Elements of $\mathbb{P}$ are elements of $\mathbf{G} \cup \{0\}$.
- A matrix A over $\mathbb{O}$ is a $\mathbb{P}$-matrix if $\det(B) \in \mathbb{P}$ for all square submatrices of A.

Previously...

Let $\mathbb{P}_1 := (\mathbb{O}_1, \mathbf{G}_1)$ and $\mathbb{P}_2 := (\mathbb{O}_2, \mathbf{G}_2)$.

- $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$ is *homomorphism* if
  - $\varphi(1) = 1$
  - $\varphi(pq) = \varphi(p)\varphi(q)$
  - If $p + q \in \mathbb{P}$ then $\varphi(p) + \varphi(q) = \varphi(p + q)$

  Example: ring homomorphism with $\varphi(\mathbf{G}_1) \subseteq \mathbf{G}_2$.
- $M(A) = M(\varphi(A))$

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- If $\mathbb{P}_1 = (\mathbb{O}_1, \mathbf{G}_1)$ and $\mathbb{P}_2 = (\mathbb{O}_2, \mathbf{G}_2)$ then

$$\mathbb{P}_1 \otimes \mathbb{P}_2 := (\mathbb{O}_1 \times \mathbb{O}_2, \mathbf{G}_1 \times \mathbf{G}_2).$$

  addition, multiplication componentwise.

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- If $M = M([I|A_1]) = M([I|A_2])$, then

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- If $M = M([I|A_1]) = M([I|A_2])$, then
$$M = M([I|A_1 \otimes A_2]).$$

$$\begin{pmatrix} 1 & 1 \\ 1 & p_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} (1,1) & (1,1) \\ (1,1) & (p_1, p_2) \end{pmatrix}$$

Previously...

Theorem 1 (Tutte 1965)**.** *Let M be a matroid. The following are equivalent:*

- *M is representable over* GF(2) *and* GF(3)
- *M is representable over $\mathbb{R}$ by a* totally unimodular matrix
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Proof. Consider partial field $\mathrm{GF}(2) \otimes \mathrm{GF}(3)$. Elements are $(0,0), (1,1), (1,-1)$. Find bijective homomorphism

$$\varphi : \mathrm{GF}(2) \otimes \mathrm{GF}(3) \to (\mathbb{Z}, \{-1, 1\}).$$

□

Today

Theorem 1' (Tutte 1965)**.** *Let M be a matroid. The following are equivalent:*

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Proof. Consider partial field $\mathrm{GF}(2) \otimes \mathbb{F}$. Elements are $(0, 0)$ and $(1, x)$ where $x \in \mathbb{F}^*$. Then ... □

Problem: $\mathrm{GF}(2) \otimes \mathbb{F}$ **is too big!**

Fundamental elements

Definition 2. *Fundamental elements of* $\mathbb{P}$

$$\mathcal{F}(\mathbb{P}) := \{p \in \mathbb{P} \mid 1 - p \in \mathbb{P}\}.$$

Theorem 3. *M is $\mathbb{P}$-representable $\Leftrightarrow$ M is $\mathbb{P}[\mathcal{F}(\mathbb{P})]$-representable.*

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$\mathcal{F}(\mathbb{P}) = \{(0,0),(1,1)\} \cup \{(1,x) \mid (1,1)-(1,x) \in \mathbb{P}\}$

$1 - (1,x) = (0, 1-x) \rightarrow x = 1.$

$\mathbb{P}[\mathcal{F}(\mathbb{P})] = \{(0,0),(1,1),(1,-1)\}$. hom. ■

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- *A' is not a scaled $\mathbb{P}'$-matrix.*
- *B is isomorphic to $A' - U$, with $|U \cap X| \leq 1, |U \cap Y| \leq 1$;*
- *If B is isomorphic to $A' - \{x, y\}$ then at least one of $A' - x, A' - y$ is 3-connected.*

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- *N is isomorphic to M'/x, $M' \backslash y$, or $M'/x\backslash y$;*
- *$N \cong M'/x\backslash y \Rightarrow M'/x$ or $M'\backslash y$ is 3-connected.*

Pf: Def. $\mathbb{P}_0 := \mathbb{P} \otimes \mathbb{P}$, $\mathbb{P}_0' := \{(x, x) \mid x \in \mathbb{P}\}$.
Apply conf. thm. to each $\mathbb{P}_0'$-rep. of N. ∎

Quinary matroids

Theorem 6. *Let M be a 3-connected matroid with at least k inequivalent representations over* GF(5).

(i) $k \geq 2 \Rightarrow M$ *representable over* $\mathbb{C}$, $\mathrm{GF}(p^2)$ *for all primes* $p \geq 3$, $\mathrm{GF}(p)$ *when* $p \equiv 1 \mod 4$.

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Ingredients of proof:

- Universal partial fields (for binary, ternary cases);
- Lift Theorem;
- Confinement Theorem.

Lift Theorem

$\widetilde{F} := \{\widetilde{p} \mid p \in \mathcal{F}(\mathbb{P})\}$ set of symbols, I ideal in $\mathbb{Z}[\widetilde{F}]$ generated by

(*i*) $\widetilde{0} - 0$; $\widetilde{1} - 1$;

(*ii*) $\widetilde{-1} + 1$ if $-1 \in \mathcal{F}(\mathbb{P})$;

(*iii*) $\widetilde{p} + \widetilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P})$, $p + q = 1$;

(*iv*) $\widetilde{p}\widetilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P})$, $pq = 1$;

(*v*) $\widetilde{p}\widetilde{q}\widetilde{r} - 1$, where $p, q, r \in \mathrm{Cr}(A)$, $pqr = 1$, and

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q^{-1} \end{bmatrix} \preceq A \in \mathcal{A}.$$

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Theorem 7. *For $A \in \mathcal{A}$, $M([I|A])$ is representable over $\mathbb{L}_{\mathcal{A}}\mathbb{P} = (\mathbb{Z}[\widetilde{F}]/I, \langle \widetilde{F} \rangle)$.*

Quinary matroids (2)

(i) $k \geq 2 \Rightarrow M$ representable over $\mathbb{C}$, GF(p^2) for all primes $p \geq 3$, GF(p) when $p \equiv 1 \mod 4$.

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Note: $\mathcal{A}$ is essential.

$$\begin{bmatrix} (1,1) & (1,1) & (1,1) \\ (1,1) & (2,2) & (3,3) \end{bmatrix}$$

Stabilizer Theorem implies these are not minors of $A \in \mathcal{A}$.

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$$\mathcal{F}(\mathbb{H}_2) = \left\{0, 1, -1, 2, \tfrac{1}{2}, i, i + 1, \tfrac{i+1}{2}, 1 - i, \tfrac{1-i}{2}, -i\right\}.$$

- Result follows by considering homomorphisms.

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- $\mathbb{H}_3' = (\mathbb{Q}(\alpha), \langle \alpha, \alpha - 1, \alpha^2 - \alpha + 1, \frac{1}{2} \rangle)$.
- $\mathbb{D} := (\mathbb{Q}, \langle -1, \frac{1}{2} \rangle)$ is induced sub-partial field. $\mathbb{D}$-confiners:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1/2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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- Consequence: matrices in $\mathcal{A}$ representable over

$$\mathbb{H}_3 := (\mathbb{Q}(\alpha), \langle \alpha, \alpha - 1, \alpha^2 - \alpha + 1 \rangle).$$

- Homomorphism to every field with an x that is no root of $\alpha, \alpha - 1, \alpha^2 - \alpha + 1$.

Quinary matroids (4,5,6)

(*iii*) $k \geq 4 \Rightarrow M$ is not binary and not ternary.

(*iv*) $k \geq 5 \Rightarrow k = 6$.

$$\mathbb{H}_4 := (\mathbb{Q}(\alpha, \beta), \langle \alpha, \beta, \alpha - 1, \beta - 1, \alpha\beta - 1, \alpha + \beta - 2\alpha\beta \rangle).$$

$$\mathbb{H}_5 := (\mathbb{Q}(\alpha, \beta, \gamma), \langle \alpha, \beta, \gamma, \alpha - 1, \beta - 1, \gamma - 1, \alpha - \gamma, \gamma - \alpha\beta, (1 - \gamma) - (1 - \alpha)\beta \rangle).$$

Main observation: *six* homomorphisms $\mathbb{H}_5 \to \mathrm{GF}(5)$.

More results

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More results

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  Theorem 8 (Whittle 1997)**.** *M 3-connected matroid representable over* GF(3) *and over* $\mathbb{F}$ *not of characteristic 3. Then at least one of these is true:*

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Proof. Consider $\mathbb{P} := \mathrm{GF}(3) \otimes \mathbb{F}$, and $\mathbb{P}$-matrix A. Then ... □

Lift Theorem

$\tilde{F} := \{\tilde{p} \mid p \in \mathcal{F}(\mathbb{P})\}$ set of symbols, I ideal in $\mathbb{Z}[\tilde{F}]$ generated by

(*i*) $\tilde{0} - 0$; $\tilde{1} - 1$;

(*ii*) $\widetilde{-1} + 1$ if $-1 \in \mathcal{F}(\mathbb{P})$;

(*iii*) $\tilde{p} + \tilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P})$, $p + q = 1$;

(*iv*) $\tilde{p}\tilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P})$, $pq = 1$;

(*v*) $\tilde{p}\tilde{q}\tilde{r} - 1$, where $p, q, r \in \mathrm{Cr}(A)$, $pqr = 1$, and

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q^{-1} \end{bmatrix} \preceq A.$$

Theorem 9. *$M([I|A])$ is representable over $\mathbb{L}_A\mathbb{P} = (\mathbb{Z}[\tilde{F}]/I, \langle\tilde{F}\rangle)$.*

Partitioning $\widetilde{F}$

(*iii*) $\widetilde{p} + \widetilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P})$, $p + q = 1$;

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Claim 10.

$$\widetilde{F} = \{0, 1\} \uplus \left\{\alpha, 1-\alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha-1}{\alpha}, \frac{1}{\alpha}\right\} \uplus \left\{\beta, 1-\beta, \frac{1}{1-\beta}, \frac{\beta}{\beta-1}, \frac{\beta-1}{\beta}, \frac{1}{\beta}\right\} \uplus \cdots$$

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Need to show:

- Relations *within* a set yield dyadic or $\sqrt[6]{1}$
- Only one such set needed

Questions, open problems

Question 11. *Can we find the forbidden minors for* GF(5)*?*

Conjecture 12. *If N is 3-connected, with universal partial field $\mathbb{P}_N$, then N stabilizes the $\mathbb{P}_N$-representable matroids.*

Question 13. *Can we classify the universal partial fields of other classes of matroids?*

First candidate: *golden ratio* matroids, i.e. those representable over $\mathrm{GF}(4) \otimes \mathrm{GF}(5)$.

The End

Thank you for your attention!

- Lifts: arXiv:0804.3263
- Confinement: arXiv:0806.4487