Confinement to sub-partial fields

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Based on joint work with Rudi Pendavingh

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Minors of matrices

Let $A$ be an $X \times Y$ matrix. A *minor* of $A$ is a matrix obtained by

- Scaling rows and columns;
- Deleting rows and columns (notation: $A - x$);
- Pivoting on a nonzero entry:

$$A = \begin{bmatrix} x' \mid c & D \end{bmatrix} \rightarrow \begin{bmatrix} x' \mid -a^{-1}c & D - a^{-1}cb \end{bmatrix} = A^{xy}$$
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- Pivoting on a nonzero entry:

$$
A = \begin{bmatrix}
\alpha & b \\
\gamma & \gamma'
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\alpha^{-1} & \alpha^{-1}b \\
-\alpha^{-1}c & D - \alpha^{-1}cb
\end{bmatrix} = A^{xy}.
$$

Correspond with minors of matroid $M([I|A])$.

$$
\begin{bmatrix}
1 & 0 \\
\vdots & A \\
0 & 1
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
A
\end{bmatrix}
$$
A partial field $\mathbb{P}$ is a pair $(\mathbb{O}, \mathbb{G})$ of a ring and a group, such that

$$-1 \in \mathbb{G} \subseteq \mathbb{O}^*.$$ 

Elements of $\mathbb{P}$ are elements of $\mathbb{G} \cup \{0\}$.

A matrix $A$ over $\mathbb{O}$ is a $\mathbb{P}$-matrix if $\det(B) \in \mathbb{P}$ for all square submatrices of $A$. 
Let $P_1 := (\emptyset_1, G_1)$ and $P_2 := (\emptyset_2, G_2)$.

- $\varphi : P_1 \rightarrow P_2$ is homomorphism if
  - $\varphi(1) = 1$
  - $\varphi(pq) = \varphi(p)\varphi(q)$
  - If $p + q \in P$ then $\varphi(p) + \varphi(q) = \varphi(p + q)$

Example: ring homomorphism with $\varphi(G_1) \subseteq G_2$.

- $M(A) = M(\varphi(A))$
• If \( P_1 = (\Omega_1, G_1) \) and \( P_2 = (\Omega_2, G_2) \) then

\[
P_1 \otimes P_2 := (\Omega_1 \times \Omega_2, G_1 \times G_2).
\]

addition, multiplication componentwise.
• If $P_1 = (\mathcal{O}_1, G_1)$ and $P_2 = (\mathcal{O}_2, G_2)$ then

$$P_1 \otimes P_2 := (\mathcal{O}_1 \times \mathcal{O}_2, G_1 \times G_2).$$

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• If $M = M([I|A_1]) = M([I|A_2])$, then

$$M = M([I|A_1 \otimes A_2]).$$
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\[
\begin{pmatrix}
1 & 1 \\
1 & \rho_1
\end{pmatrix} \otimes \begin{pmatrix}
1 & 1 \\
1 & \rho_2
\end{pmatrix} = \begin{pmatrix}
(1, 1) & (1, 1) \\
(1, 1) & (\rho_1, \rho_2)
\end{pmatrix}
\]
**Theorem 1** (Tutte 1965). Let $M$ be a matroid. The following are equivalent:

- $M$ is representable over $\text{GF}(2)$ and $\text{GF}(3)$
- $M$ is representable over $\mathbb{R}$ by a totally unimodular matrix
- $M$ is representable over every field
Theorem 1 (Tutte 1965). Let \( M \) be a matroid. The following are equivalent:

- \( M \) is representable over \( GF(2) \) and \( GF(3) \)
- \( M \) is representable over \( \mathbb{R} \) by a totally unimodular matrix
- \( M \) is representable over every field

Proof. Consider partial field \( GF(2) \otimes GF(3) \). Elements are \( (0, 0), (1, 1), (1, -1) \). Find bijective homomorphism

\[ \varphi : GF(2) \otimes GF(3) \to (\mathbb{Z}, \{-1, 1\}) . \]
Theorem 1’ (Tutte 1965). Let $M$ be a matroid. The following are equivalent:

- $M$ is representable over $\text{GF}(2)$ and field not of characteristic 2
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**Theorem 1’** (Tutte 1965). Let $M$ be a matroid. The following are equivalent:

- $M$ is representable over $\text{GF}(2)$ and field not of characteristic 2
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**Proof.** Consider partial field $\text{GF}(2) \otimes F$. Elements are $(0, 0)$ and $(1, x)$ where $x \in F^*$. Then ...

**Problem:** $\text{GF}(2) \otimes F$ is too big!
**Definition 2.** *Fundamental elements of $\mathbb{P}$*

\[ \mathcal{F}(\mathbb{P}) := \{ p \in \mathbb{P} \mid 1 - p \in \mathbb{P} \}. \]

**Theorem 3.** *$M$ is $\mathbb{P}$-representable $\iff M$ is $\mathbb{P}[\mathcal{F}(\mathbb{P})]$-representable.*
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**Proof.** Consider partial field $\text{GF}(2) \otimes F$. Elements are $(0, 0)$ and $(1, x)$ where $x \in F^*$. Then ...

$$F(P) = \{(0,0), (1,1)\} \cup \{ (1, x) \mid (1)-(1, x) \in P \}$$

$$P[F(P)] = \{(0,0), (1,1), (1, -1)\}$$, hom. \(\square\)
Theorem 4. $\mathcal{P}' \subseteq \mathcal{P}$ induced. $B$ 3-connected scaled $\mathcal{P}'$-matrix. A 3-connected $\mathcal{P}$-matrix with submatrix $B$. Exactly one of these is true:

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- \( A' \) is not a scaled \( P' \)-matrix.
- \( B \) is isomorphic to \( A' - U \), with \(|U \cap X| \leq 1, |U \cap Y| \leq 1| \).
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- \( A' \) is not a scaled \( P' \)-matrix.
- \( B \) is isomorphic to \( A' - U \), with \( |U \cap X| \leq 1, |U \cap Y| \leq 1 \);
- If \( B \) is isomorphic to \( A' - \{x, y\} \) then at least one of \( A' - x, A' - y \) is 3-connected.
Stabilizer Theorem

Matroid $N$ stabilizes $M$ over $\mathbb{P}$ if the representation of $N$ determines uniquely that of $M$. 
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- $N$ does not stabilize $M'$;
- $N$ is isomorphic to $M'/x, M'\setminus y$, or $M'/x\setminus y$;
- If $N \cong M'/x\setminus y$ then one of $M'/x, M'\setminus y$ is 3-connected.
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- $N$ does not stabilize $M'$;
- $N$ is isomorphic to $M'/x, M'/\backslash y$, or $M'/x\backslash y$;
- $N \cong M'/x\backslash y \Rightarrow M'/x$ or $M'/\backslash y$ is 3-connected.

Proof: Def. $P_0 := P \otimes P$, $P'_0 := \{(x, x) | x \in P^5\}$

Apply conf. thm. to each $P'_0$-rep. of $N$. 

/ department of mathematics and computer science
Theorem 6. Let $M$ be a 3-connected matroid with at least $k$ inequivalent representations over $GF(5)$.

(i) $k \geq 2 \Rightarrow M$ representable over $\mathbb{C}$, $GF(p^2)$ for all primes $p \geq 3$, $GF(p)$ when $p \equiv 1 \pmod{4}$. 

Quinary matroids
Theorem 6. Let $M$ be a 3-connected matroid with at least $k$ inequivalent representations over $GF(5)$.

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(ii) $k \geq 3 \Rightarrow M$ representable over every field with at least five elements.
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(iii) $k \geq 4 \Rightarrow M$ is not binary and not ternary.
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(iv) $k \geq 5 \Rightarrow k = 6$. 
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Ingredients of proof:

- Universal partial fields (for binary, ternary cases);
- Lift Theorem;
- Confinement Theorem.
$\tilde{F} := \{\tilde{p} \mid p \in \mathcal{F}(\mathbb{P})\}$ set of symbols, $I$ ideal in $\mathbb{Z}[\tilde{F}]$ generated by

(i) $\tilde{0} - 0; \tilde{1} - 1$;

(ii) $\tilde{-1} + 1$ if $-1 \in \mathcal{F}(\mathbb{P})$;

(iii) $\tilde{p} + \tilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P}), p + q = 1$;

(iv) $\tilde{p}\tilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P}), pq = 1$;

(v) $\tilde{p}\tilde{q}\tilde{r} - 1$, where $p, q, r \in \text{Cr}(A), pqr = 1$, and

\[
\begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q^{-1} \end{bmatrix} \preceq A \in \mathcal{A}.
\]
Lift Theorem

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(iv) $\tilde{p}\tilde{q} - 1$, where $p, q \in \mathcal{F}(P), pq = 1$;
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\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & p & q^{-1}
\end{bmatrix} \preceq A \in A.
\]

**Theorem 7.** For $A \in A$, $M([I|A])$ is representable over $\mathbb{L}_A^P = (\mathbb{Z}[\tilde{F}]/I, \langle \tilde{F} \rangle)$. 

/ department of mathematics and computer science
(i) \( k \geq 2 \Rightarrow M \) representable over \( \mathbb{C}, \text{GF}(p^2) \) for all primes \( p \geq 3 \), \( \text{GF}(p) \) when \( p \equiv 1 \mod 4 \).
(i) $k \geq 2 \Rightarrow M$ representable over $\mathbb{C}$, $GF(p^2)$ for all primes $p \geq 3$, $GF(p)$ when $p \equiv 1 \mod 4$.

- $P := GF(5) \otimes GF(5)$;
Quinary matroids (2)

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- $\mathcal{P} := \text{GF}(5) \otimes \text{GF}(5)$;
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- $\mathcal{A}$ is set of $\mathcal{P}$-matrices $A$ with $\varphi_1(A) \not\sim \varphi_2(A)$;
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- $H_2 := L_\mathcal{A}\mathcal{P}$. 

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- $H_2 := L_{\mathcal{A}} \mathcal{P}$.

Note: $\mathcal{A}$ is essential.

$$
\begin{pmatrix}
(1, 1) & (1, 1) & (1, 1) \\
(1, 1) & (2, 2) & (3, 3)
\end{pmatrix}
$$

Stabilizer Theorem implies these are not minors of $A \in \mathcal{A}$. 
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- $\varphi_i : \mathbb{P} \rightarrow GF(5)$: projection on $i$th coordinate;
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$$H_2 = (\mathbb{C}, \langle i, 1 - i \rangle).$$
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- \( \mathcal{A} \) is set of \( \mathbb{P} \)-matrices \( A \) with \( \varphi_1(A) \not\sim \varphi_2(A) \);
- \( \mathcal{H}_2 := \mathbb{L}_\mathcal{A}\mathbb{P} \).

\[
\mathcal{H}_2 = (\mathbb{C}, \langle i, 1 - i \rangle).
\]

\[
\mathcal{F}(\mathcal{H}_2) = \left\{ 0, 1, -1, 2, \frac{1}{2}, i, i + 1, \frac{i+1}{2}, 1 - i, \frac{1-i}{2}, -i \right\}.
\]

- Result follows by considering homomorphisms.
(ii) $k \geq 3 \Rightarrow M$ representable over every field with at least five elements.
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- \( \mathcal{A} \) is set of \( \mathcal{P} \)-matrices \( A \) with \( \varphi_i(A) \) nonequivalent;
- \( H'_3 := L_{\mathcal{A}} \mathcal{P} \).
(ii) $k \geq 3 \Rightarrow M$ representable over every field with at least five elements.

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- $H'_3 := L_{\mathcal{A}} \mathcal{P}$.

$$H'_3 = (\mathbb{Q}(\alpha), \langle \alpha, \alpha - 1, \alpha^2 - \alpha + 1, \frac{1}{2} \rangle).$$
(ii) $k \geq 3 \Rightarrow M$ representable over every field with at least five elements.

- $H'_3 = (\mathbb{Q}(\alpha), \langle \alpha, \alpha - 1, \alpha^2 - \alpha + 1, \frac{1}{2} \rangle)$.
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- $H_3' = (\mathbb{Q}(\alpha), \langle \alpha, \alpha - 1, \alpha^2 - \alpha + 1, \frac{1}{2} \rangle)$.

- $D := (\mathbb{Q}, \langle -1, \frac{1}{2} \rangle)$ is induced sub-partial field. $D$-confiners:

  $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1/2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
(ii) $k \geq 3 \Rightarrow M$ representable over every field with at least five elements.

- $\mathcal{H}_3' = (\mathbb{Q}(\alpha), \langle \alpha, \alpha - 1, \alpha^2 - \alpha + 1, \frac{1}{2} \rangle)$.

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$$
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\end{bmatrix},
\begin{bmatrix}
1 & 1 \\
1 & 1/2 \\
\end{bmatrix},
\begin{bmatrix}
1 & 1 \\
1 & -1 \\
\end{bmatrix}
$$

- Consequence: matrices in $\mathcal{A}$ representable over $\mathcal{H}_3 := (\mathbb{Q}(\alpha), \langle \alpha, \alpha - 1, \alpha^2 - \alpha + 1 \rangle)$.

- Homomorphism to every field with an $x$ that is no root of $\alpha, \alpha - 1, \alpha^2 - \alpha + 1$. 
(iii) $k \geq 4 \Rightarrow M$ is not binary and not ternary.
(iv) $k \geq 5 \Rightarrow k = 6$.

$H_4 := (\mathbb{Q}(\alpha, \beta), \langle \alpha, \beta, \alpha - 1, \beta - 1, \alpha\beta - 1, \alpha + \beta - 2\alpha\beta \rangle)$.

$H_5 := (\mathbb{Q}(\alpha, \beta, \gamma), \langle \alpha, \beta, \gamma, \alpha - 1, \beta - 1, \gamma - 1, \alpha - \gamma, \gamma - \alpha\beta, (1 - \gamma) - (1 - \alpha)\beta \rangle)$.

Main observation: six homomorphisms $H_5 \rightarrow \text{GF}(5)$.
More results

Corollaries of Confinement Theorem:

- Whittle’s Stabilizer Theorem;
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- Whittle’s characterization of ternary matroids:

**Theorem 8** (Whittle 1997). $M$ 3-connected matroid representable over GF(3) and over $\mathbb{F}$ not of characteristic 3. Then at least one of these is true:

(i) $M$ is dyadic;
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  (i) $M$ is dyadic;
  (ii) $M$ is sixth-roots-of-unity.

  **Proof.** Consider $\mathbb{P} := \text{GF}(3) \otimes \mathbb{F}$, and $\mathbb{P}$-matrix $A$. Then ...
\[ \tilde{F} := \{ \tilde{p} \mid p \in \mathcal{F}(\mathbb{P}) \} \] set of symbols, \( I \) ideal in \( \mathbb{Z}[\tilde{F}] \) generated by

(i) \( \tilde{0} - 0; \tilde{1} - 1 \);

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(iv) \( \tilde{p}\tilde{q} - 1 \), where \( p, q \in \mathcal{F}(\mathbb{P}) \), \( pq = 1 \);

(v) \( \tilde{p}\tilde{q}\tilde{r} - 1 \), where \( p, q, r \in \text{Cr}(A) \), \( pqr = 1 \), and

\[
\begin{bmatrix}
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\]

**Theorem 9.** \( M([I|A]) \) is representable over \( \mathbb{L}_{A^P} = (\mathbb{Z}[\tilde{F}]/I, \langle \tilde{F} \rangle) \).
(iii) \( \tilde{q} + \tilde{p} - 1 \), where \( p, q \in \mathcal{F}(\mathbb{P}) \), \( p + q = 1 \);
(iv) \( \tilde{p}\tilde{q} - 1 \), where \( p, q \in \mathcal{F}(\mathbb{P}) \), \( pq = 1 \);

Claim 10.

\[
\tilde{F} = \{0, 1\} \cup \left\{ \alpha, 1 - \alpha, \frac{1}{1 - \alpha'}, \frac{\alpha}{\alpha - 1}, \frac{\alpha - 1}{\alpha}, 1 \right\} \cup \left\{ \beta, 1 - \beta, \frac{1}{1 - \beta'}, \frac{\beta}{\beta - 1}, \frac{\beta - 1}{\beta}, 1 \right\} \cup \ldots
\]
(iii) \( \tilde{p} + \tilde{q} - 1 \), where \( p, q \in \mathcal{F}(\mathbb{P}) \), \( p + q = 1 \);
(iv) \( \tilde{p}\tilde{q} - 1 \), where \( p, q \in \mathcal{F}(\mathbb{P}) \), \( pq = 1 \);

**Claim 10.**

\[
\tilde{F} = \{0, 1\} \cup \left\{ \alpha, 1 - \alpha, \frac{1}{1 - \alpha'}, \frac{\alpha}{\alpha - 1}, \frac{\alpha - 1}{\alpha}, 1 \right\} \cup \left\{ \beta, 1 - \beta, \frac{1}{1 - \beta'}, \frac{\beta}{\beta - 1}, \frac{\beta - 1}{\beta}, 1 \right\} \cup \ldots
\]

Need to show:
- Relations *within* a set yield dyadic or \( \sqrt[6]{1} \)
- Only one such set needed
Question 11. Can we find the forbidden minors for GF(5)?

Conjecture 12. If $N$ is 3-connected, with universal partial field $\mathbb{P}_N$, then $N$ stabilizes the $\mathbb{P}_N$-representable matroids.

Question 13. Can we classify the universal partial fields of other classes of matroids?

First candidate: golden ratio matroids, i.e. those representable over $\text{GF}(4) \otimes \text{GF}(5)$. 
The End

Thank you for your attention!

- Lifts: arXiv:0804.3263
- Confinement: arXiv:0806.4487