Cross-ratios in partial fields

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Based on joint work with Rudi Pendavingh

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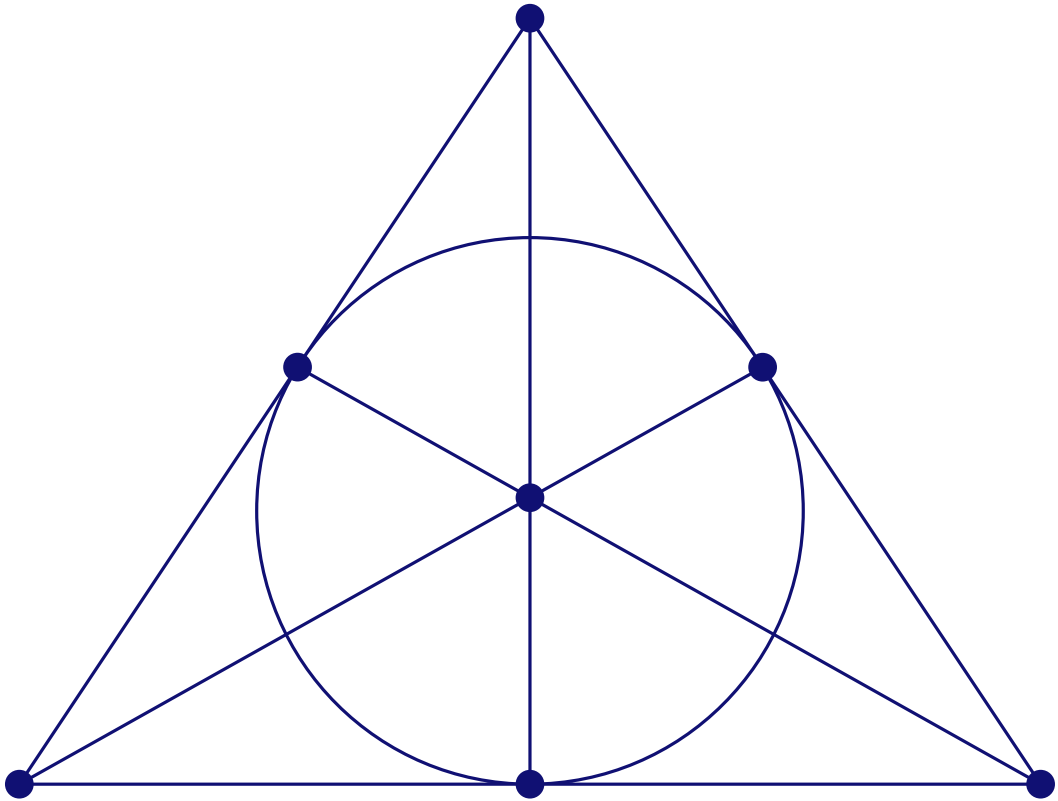
Partial fields

Today's topic

Matroids



Today's topic

Cross-ratios

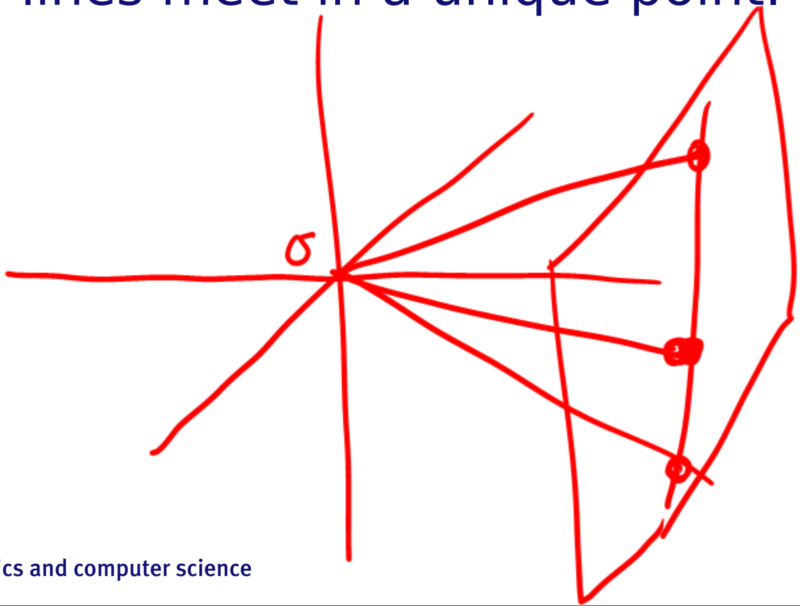
$$\frac{3}{2}$$

Projective geometry

Projective n-space:

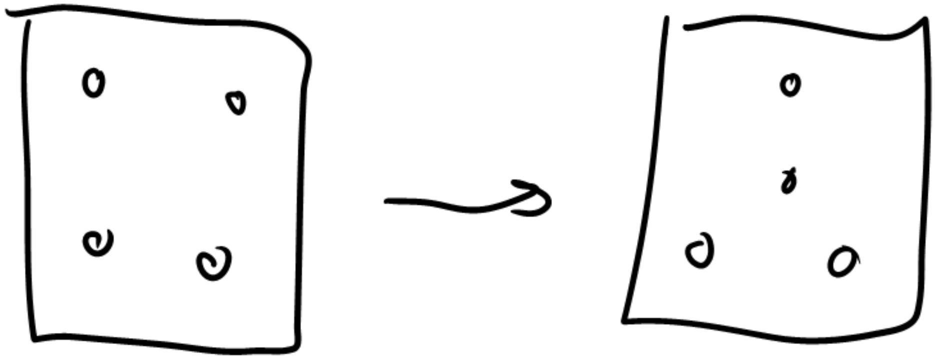
- “points” are 1-dimensional subspaces of $\mathbb{F}^{n+1}$;
- “lines” are 2-dimensional subspaces;
- ...

Through every 2 points is a unique line; every 2 coplanar lines meet in a unique point.



Projective geometry

- Two subspaces are “incident” if one is contained in the other.
- Projective transformations: preserve incidences.
- Projective transformations are *invertible linear transformations*.
- Projective transformations do not preserve angles, distances, “between-ness”.



Cross-ratio

Cross-ratio: quantity that *is* preserved under linear transformations.

$$\mathrm{Cr}(A,B,C,D) := \frac{AC \cdot DB}{CB \cdot AD}$$

Scale so that $C = A + B$; $D = \alpha A + B$. Then $\mathrm{Cr}(A,B,C,D) = \alpha$.

$$\begin{array}{cccc} A & B & C & D \end{array} \qquad \begin{array}{cccc} A & B & D & C \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & \alpha^{-1} \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{cccc} A & C & B & D \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1-\alpha \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

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Scale so that $C = A + B$; $D = \alpha A + B$. Then $\mathrm{Cr}(A,B,C,D) = \alpha$.

Changing the order changes the cross ratio.

$$\left\{\alpha, 1-\alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha-1}{\alpha}, \frac{1}{\alpha}\right\}$$

Finite configurations

What we will look at:

- Finite set E of points in projective n-space over $\mathbb{F}$;
- If $\mathcal{B}$ is the collection of minimal spanning subsets of a configuration E, then we call $M = (E, \mathcal{B})$ a combinatorial geometry or *matroid*.
- If E is represented as the set of columns of a matrix A, then we write $M(A)$ for this matroid.

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What cross-ratios can we find?

- Restriction to subsets of points;
- Projection from $v \in E$ onto orthogonal subspace.

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Minors of matrices

Let A be an $X \times Y$ matrix.

$$\begin{array}{c|c} X & Y \\ \left[\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}\right. & \left. A \right] \end{array} \longleftrightarrow X \overset{Y}{\left[\quad A \quad \right]}$$

Minors of matrices

Let A be an $X \times Y$ matrix.

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{c} X \qquad\qquad Y \\ \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & A \\ 0 & & 1 & \end{array}\right] \end{array} \longleftrightarrow X \begin{array}{c} Y \\ \left[\begin{array}{c} \\ A \\ \\ \end{array}\right] \end{array}$$

A *minor* of A is a matrix obtained by

- Scaling rows and columns;
- Deleting rows and columns (notation: $A - x$);
- Pivoting on a nonzero entry:

$$A = \begin{array}{c} \\ x \\ X' \end{array} \begin{array}{c} \;\; y \quad Y' \\ \left[\begin{array}{c|c} a & b \\ \hline c & D \end{array}\right] \end{array} \rightarrow \begin{array}{c} \\ y \\ X' \end{array} \begin{array}{c} x \qquad\qquad Y' \\ \left[\begin{array}{c|c} a^{-1} & a^{-1}b \\ \hline -a^{-1}c & D - a^{-1}cb \end{array}\right] \end{array} = A^{xy}.$$

Cross-ratios of a matrix

A *minor* of A is a matrix obtained by

- Scaling rows and columns;
- Deleting rows and columns;
- Pivoting on a nonzero entry:

Notation: $B \preceq A$.

$$\mathrm{Cr}(A) := \left\{ x \mid \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \preceq A \right\}$$

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Let $G := \langle -1, \mathrm{Cr}(A) \rangle$ be the multiplicative subgroup of $\mathbb{F}$ generated by $\{-1\} \cup \mathrm{Cr}(A)$.

Lemma 1. *There exist nonsingular diagonal matrices D_1, D_2 such that every subdeterminant of $D_1 A D_2$ is in G.*

Partial fields

Idea (Semple, Whittle 1996): throw away all unnecessary field structure!

Definition 2.

A partial field $\mathbb{P}$ *is a pair* $(\mathbb{O}, \mathbf{G})$ *of a ring and a group, such that* $-1 \in \mathbf{G} \subseteq \mathbb{O}^*$.
Elements of $\mathbb{P}$ *are elements of* $\mathbf{G} \cup \{0\}$.

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$$p + q = p(1 - (-\frac{q}{p}))$$

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A matrix A over $\mathbb{O}$ is a $\mathbb{P}$-matrix if $\det(B) \in \mathbb{P}$ for all square submatrices of A.

Open problem

Problem 3. *Let* $\mathbb{P} = (\mathbb{O}, \mathbf{G})$ *be a partial field.*

(i) For $p \in \mathbb{P}$*, decide if* $1 - p \in \mathbb{P}$*;*

(ii) Find $\{p \in \mathbb{P} \mid 1 - p \in \mathbb{P}\}$*.*

$(\mathbb{Q}, \langle -1, 2, 3 \rangle)$ $\qquad \pm 2^x 3^y$

sol^s to $\pm 2^x \pm 3^y = 1$

Thm: only one nontriv. sol:

$3^2 - 2^3 = 1$

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Special case: $\mathbb{O} = \mathbb{Z}[x_1, \ldots, x_n]/I$, I finitely generated, and $\mathbf{G} = \langle x_1, \ldots, x_n \rangle$.

Homomorphisms

Let $\mathbb{P}_1 := (\mathbb{O}_1, \mathbf{G}_1)$ and $\mathbb{P}_2 := (\mathbb{O}_2, \mathbf{G}_2)$.

- $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$ is *homomorphism* if
  - $\varphi(1) = 1$
  - $\varphi(pq) = \varphi(p)\varphi(q)$
  - If $p + q \in \mathbb{P}$ then $\varphi(p) + \varphi(q) = \varphi(p + q)$

  Example: ring homomorphism with $\varphi(\mathbf{G}_1) \subseteq \mathbf{G}_2$.
- $M([I|A]) = M(\varphi([I|A]))$

Examples

- "Trivial" example: A is $\mathbb{R}$-matrix, $\mathrm{Cr}(A) = \{0, 1\}$.
- Then A is (up to scaling), a $\mathbb{U}_0$-matrix, where $\mathbb{U}_0 := (\mathbb{Z}, \{-1, 1\})$.
- Also known as *totally unimodular matrix*
- Homomorphism to *every other field!*

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- $\mathbb{F} = \mathbb{R}$, $\mathrm{Cr}(A) = \{0, 1, -1, 2, \frac{1}{2}\}$.
- Then A is (up to scaling), a $\mathbb{D}$-matrix, where $\mathbb{D} := (\mathbb{Z}[\frac{1}{2}], \langle -1, 2\rangle)$.
- Homomorphism to *every field not of characteristic 2*.

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The other way round

Regular: $\mathbb{U}_0 := (\mathbb{Z}, \{-1, 1\})$.

Theorem 4 (Tutte 1965)**.** *Let M be a matroid. The following are equivalent:*

- *M is representable over* GF(2) *and* GF(3)

$\Downarrow$

- *M is representable over* $\mathbb{R}$ *by a* totally unimodular matrix

$\Downarrow$

- *M is representable over every field*

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Products of partial fields

- If $\mathbb{P}_1 = (\mathbb{O}_1, \mathbf{G}_1)$ and $\mathbb{P}_2 = (\mathbb{O}_2, \mathbf{G}_2)$ then

$$\mathbb{P}_1 \otimes \mathbb{P}_2 := (\mathbb{O}_1 \times \mathbb{O}_2, \mathbf{G}_1 \times \mathbf{G}_2).$$

addition, multiplication componentwise.

$(0,0), (x,y) \quad x,y \neq 0$

$(0,0), (1,1), (x,y) \quad x,y \neq 0,1$

$\left.\begin{array}{l} 1-x \\ 1-y \end{array}\right\}$

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- If $M = M([I|A_1]) = M([I|A_2])$, then

$$M = M([I|A_1 \otimes A_2]).$$

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addition, multiplication componentwise.
- If $M = M([I|A_1]) = M([I|A_2])$, then
$$M = M([I|A_1 \otimes A_2]).$$

$$\begin{pmatrix} 1 & 1 \\ 1 & p_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & p_2 \end{pmatrix} = \begin{pmatrix} (1,1) & (1,1) \\ (1,1) & (p_1, p_2) \end{pmatrix}$$

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Theorem 5 (Tutte 1965)**.** *Let M be a matroid. The following are equivalent:*

- *M is representable over* GF(2) *and* GF(3)
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Proof. Consider partial field $\mathrm{GF}(2) \otimes \mathrm{GF}(3)$. Elements are $(0,0), (1,1), (1,-1)$. Find bijective homomorphism

$$\varphi : \mathrm{GF}(2) \otimes \mathrm{GF}(3) \to (\mathbb{Z}, \{-1, 1\}).$$

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The other way round

Dyadic: $\mathbb{D} := (\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle)$.

Theorem 6 (Whittle 1997)**.** *Let M be a matroid. The following are equivalent:*

- *M is representable over* GF(3) *and* GF(5)
- *M is* $\mathbb{D}$*-representable*
- *M is representable over every field not of characteristic 2*

Problem: GF(3) $\otimes$ GF(5) is finite, but $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle)$ is infinite!

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Problem: $\mathrm{GF}(3) \otimes \mathrm{GF}(5)$ is finite, but $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \langle -1, 2\rangle)$ is infinite!

$$\mathrm{Cr}(\mathrm{GF}(3) \otimes \mathrm{GF}(5)) = \{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix} \}$$

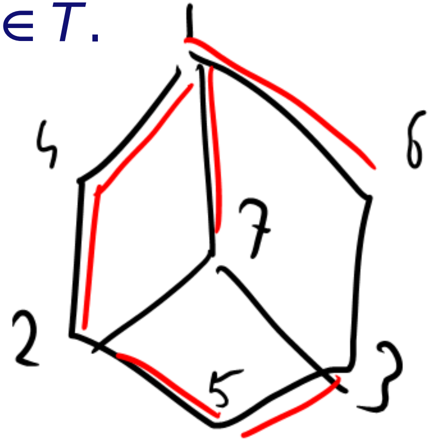
$$\mathrm{Cr}(\mathbb{D}) = \{ 0, 1, 2, \frac{1}{2}, -1 \}$$

Normalization

Choose basis X. A is an $X \times Y$ rep. matrix over $\mathbb{F}$.

- Positions of zeroes are fixed.
- $G(A)$ is bipartite graph with vertex classes X, Y. $xy \in E(G)$ if and only if $A_{xy} \neq 0$.
- Suppose edges $x_1y_1, \ldots, x_ky_k$ form spanning forest of $G(A)$. Let $\theta_1, \ldots, \theta_k \in \mathbb{F}^*$. Can scale rows and columns so that $A_{x_iy_i} = \theta_i$ for all i.
- A is *normalized* if, for some spanning forest T, $A_{xy} = 1$ for all $xy \in T$.

	4	5	6	7
1	1	0	1	1
2	1	1	0	1
3	0	1	−1	1



The Lift Theorem

A recipe

Input:

- Partial fields $\mathbb{P}, \widehat{\mathbb{P}}$
- Bijection $^{\uparrow} : \mathrm{Cr}(\mathbb{P}) \to \mathrm{Cr}(\widehat{\mathbb{P}})$
- $\mathbb{P}$-matrix A

Output: matrix $A^{\uparrow}$ with entries in $\widehat{\mathbb{P}}$

$$A = \begin{bmatrix} 1 & 0 & \binom{2}{4} & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & \binom{2}{2} \end{bmatrix} \qquad A^{\uparrow} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

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The Lift Theorem

Theorem 7 (vZ, Pendavingh 2008)**.** *Let* $\mathbb{P}, \widehat{\mathbb{P}}, \uparrow$ *be as before and let* A *be a* $\mathbb{P}$*-matrix. Then either*

- $A^\uparrow$ *is a* $\widehat{\mathbb{P}}$*-matrix and* $M([I|A]) = M([I|A^\uparrow])$*, or*
- $B^\uparrow$ *is not a* $\mathbb{P}$*-matrix for some*

$$B \in \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q \end{bmatrix} \mid p, q \in \mathrm{Cr}(\mathbb{P}) \right\} \cup \left\{ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \right\}.$$

Applications

Dyadic: $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle)$.

Theorem 8 (Whittle 1997)**.** *Let M be a matroid. The following are equivalent:*

(i) M is representable over $\mathrm{GF}(3) \otimes \mathrm{GF}(5)$

(ii) M is $\mathbb{D}$-representable

(iii) M is representable over every field not of characteristic 2

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Applications

Near-regular: $\mathbb{U}_1 := (\mathbb{Q}(\alpha), \langle -1, \alpha, 1-\alpha \rangle)$.

Theorem 9 (Whittle 1997)**.** *Let M be a matroid. The following are equivalent:*

(i) M is representable over $\mathrm{GF}(3) \otimes \mathrm{GF}(4) \otimes \mathrm{GF}(5)$

(ii) M is representable over $\mathrm{GF}(3) \otimes \mathrm{GF}(8)$

(iii) M is $\mathbb{U}_1$-representable

(iv) M is representable over every partial field with at least 3 elements.

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Applications

Golden ratio: $\mathbb{G} := (\mathbb{Q}(\sqrt{5}), \langle -1, \tau \rangle)$ where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the *golden ratio*, i.e. a root of $x^2 - x - 1$. Then $\langle -1, \tau \rangle$ are the units of the *ring of integers* of $\mathbb{Q}(\sqrt{5})$.

Theorem 10 (Vertigan)**.** *Let M be a matroid. The following are equivalent:*

(i) M is representable over GF(4) ⊗ GF(5)*;*

(ii) M is $\mathbb{G}$*-representable;*

(iii) M is representable over GF(5)*, over* GF(p^2) *for all primes p, and over* GF(p) *when* $p \equiv \pm 1 \mod 5$.

Applications

Gaussian: $\mathbb{H}_2 := (\mathbb{C}, \langle i, 1-i \rangle)$, where i is a root of $x^2 + 1 = 0$.

Theorem 11. *Let M be a 3-connected matroid with a $U_{2,5}$- or $U_{3,5}$-minor. The following are equivalent:*

(i) M has 2 inequivalent representations over $\mathrm{GF}(5)$*;*

(ii) M is $\mathbb{H}_2$-representable;

(iii) M has two inequivalent representations over $\mathrm{GF}(5)$*, is representable over* $\mathrm{GF}(p^2)$ *for all primes $p \geq 3$, and over* $\mathrm{GF}(p)$ *when $p \equiv 1 \mod 4$.*

Open problems

Conjecture 12. *A matroid is representable over* $GF(2^k)$ *for all* $k > 1$ *if and only if it is representable over* $\mathbb{U}_1^{(2)}$.

Here $\mathbb{U}_1^{(2)} = (GF(2)(\alpha), \langle \alpha, 1+\alpha \rangle)$.

Question 13. *Which partial fields* $\mathbb{P}$ *are such that whenever the set of* $\mathbb{P}$*-representable matroids is also representable over a field* $\mathbb{F}$, *there exists a homomorphism* $\varphi : \mathbb{L}\mathbb{P} \to \mathbb{F}$?

Question 14. *Can more results be derived from the Lift Theorem?*

An algebraic construction

$\widetilde{F} := \{\widetilde{p} \mid p \in \mathcal{F}(\mathbb{P})\}$ set of symbols, I ideal in $\mathbb{Z}[\widetilde{F}]$ generated by

(*i*) $\widetilde{0} - 0$; $\widetilde{1} - 1$;

(*ii*) $\widetilde{-1} + 1$ if $-1 \in \mathcal{F}(\mathbb{P})$;

(*iii*) $\widetilde{p} + \widetilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P})$, $p + q = 1$;

(*iv*) $\widetilde{p}\widetilde{q} - 1$, where $p, q \in \mathcal{F}(\mathbb{P})$, $pq = 1$;

(*v*) $\widetilde{p}\widetilde{q}\widetilde{r} - 1$, where $p, q, r \in \mathrm{Cr}(A)$, $pqr = 1$, and

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q^{-1} \end{bmatrix} \preceq A \in \mathcal{A}.$$

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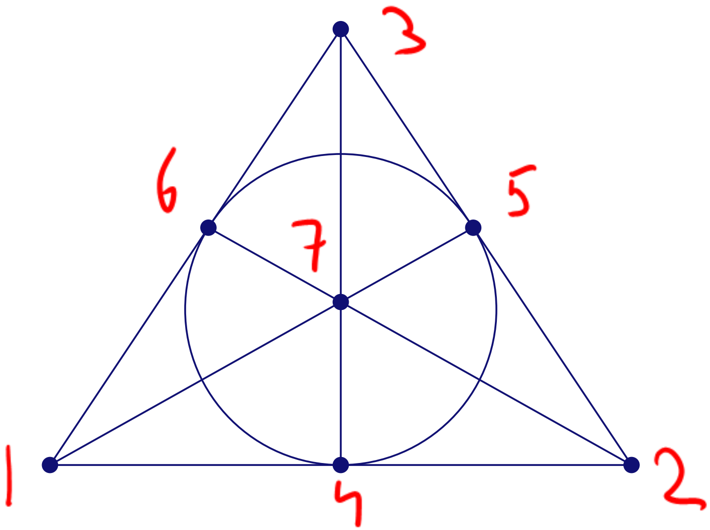
(*v*) $\widetilde{p}\widetilde{q}\widetilde{r} - 1$, where $p, q, r \in \mathrm{Cr}(A)$, $pqr = 1$, and

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & p & q^{-1} \end{bmatrix} \preceq A \in \mathcal{A}.$$

Theorem 15. *For $A \in \mathcal{A}$, $M([I|A])$ is representable over $\mathbb{L}_{\mathcal{A}}\mathbb{P} = (\mathbb{Z}[\widetilde{F}]/I, \langle \widetilde{F} \rangle)$.*

Universal partial field



1	2	3	4	5	6	7
1	0	0	1	0	1	1
0	1	0	1	1	0	y
0	0	1	0	1	x	z

$\{1,5,7\} \rightsquigarrow y = z$ $\quad \{2,6,7\} \rightsquigarrow x = z$ $\quad \{3,4,7\} \rightsquigarrow y = 1$

$\{4,5,6\} \rightsquigarrow z \equiv 0$

Universal representation

$M = (X \cup Y, \mathcal{B})$; X basis. $A_{M,X} = (a_{ij})$ $X \times Y$ matrix; a_{ij} unknowns. For each $B \in \mathcal{B}$ an unknown i_B. Ring $\mathbb{Z}[\{a_{ij}\} \cup \{i_B\}]$. Construct ideal I:

- If $(X \setminus i) \cup j \notin \mathcal{B}$ then $a_{ij} \in I$;
- T spanning forest of $G(A)$. If $ij \in T$ then $a_{ij} - 1 \in I$;
- $Z \subseteq X \cup Y$, $|Z| = r$. If $Z \notin \mathcal{B}$ then $\det(A_{M,X}[Z]) \in I$;
- $Z \subseteq X \cup Y$, $|Z| = r$. If $Z \in \mathcal{B}$ then $\det(A_{M,X}[Z]) i_Z - 1 \in I$.

$$\mathbb{P}_M := (\mathbb{Z}[\{a_{ij}\} \cup \{i_B\}]/I, \mathrm{Cr}(A_{M,X})).$$

Universal representation

Theorem 16. *If A is an $X \times Y$ matrix over field $\mathbb{F}$ such that $M = M([I|A])$, and A is T-normalized, then there is a ring homomorphism $\varphi : \overline{\mathbb{B}}_M \to \mathbb{F}$ such that*

$$\varphi(A_{M,X}) = A.$$

Theorem 17. *$\overline{\mathbb{B}}_M$ does not depend on choice of X or T (different choices give isomorphic rings).*

Settlement Theorem

Matroid N *settles* M if $\mathbb{P}_N \rightarrow \mathbb{P}_M$ is surjective.

Theorem 18 (Pendavingh, Van Zwam 2008)**.** *M, N 3-connected matroids, $N \preceq M$. Exactly one of the following is true:*

(i) N settles M;

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(i) N settles M;

(ii) M has 3-connected minor M' such that

- *N does not settle M';*

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(i) N settles M;

(ii) M has 3-connected minor M' such that

- *N does not settle M';*
- *N is isomorphic to M'/x, $M' \backslash y$, or $M'/x \backslash y$;*
- *If $N \cong M'/x \backslash y$ then one of $M'/x, M' \backslash y$ is 3-connected.*

Very useful: limits number of representations of M.

$$\mathbb{P}_N \to \mathbb{P}_M \to \mathbb{P}.$$

Examples

The regular partial field

$$\mathbb{U}_0 = (\mathbb{Z}, \{-1, 1\})$$

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Examples

The near-regular partial field

$$\mathbb{U}_1 = (\mathbb{Q}(\alpha), \langle -1, \alpha, 1-\alpha \rangle)$$

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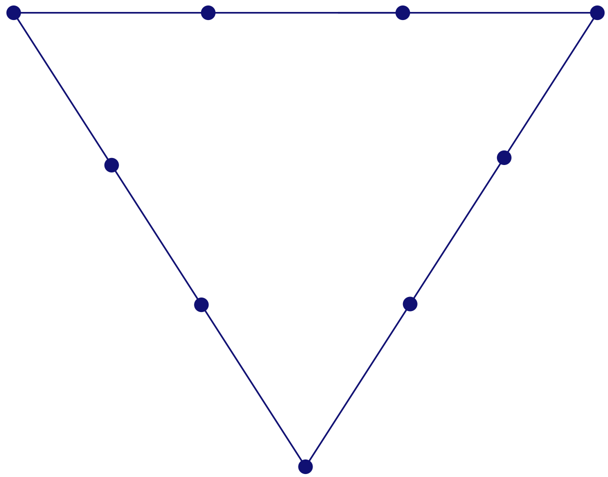
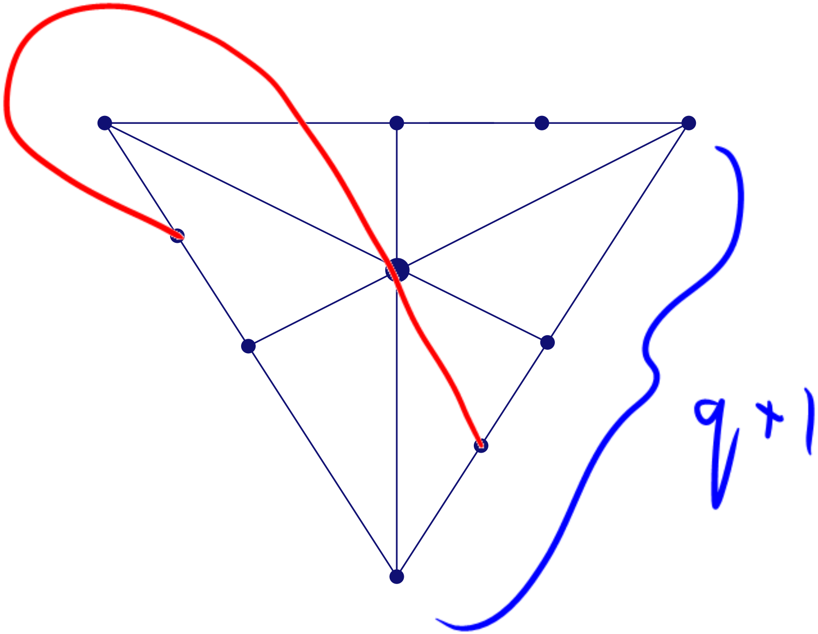
The dyadic partial field



$$\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle)$$

Examples

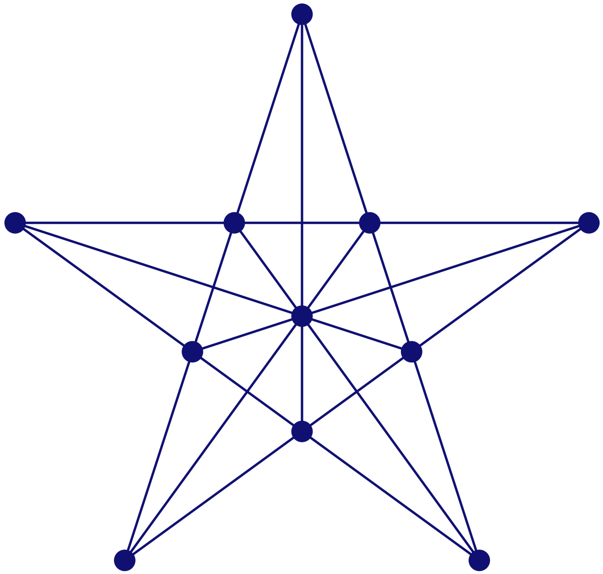
GF(q)



$$GF(q) = (GF(q), GF(q)^*)$$

Examples

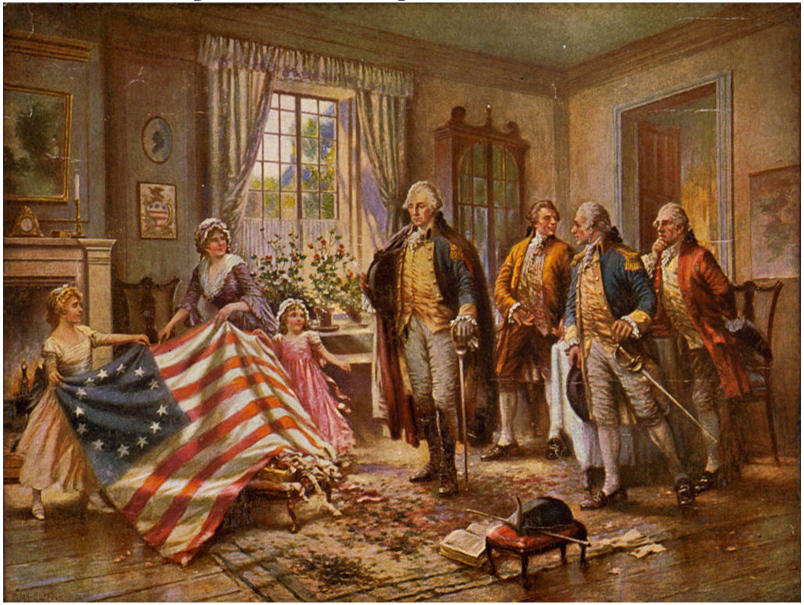
The golden ratio partial field



$$\mathbb{D} = (\mathbb{Q}(\sqrt{5}), \langle -1, \tau \rangle)$$

The End

Thank you for your attention!



- Lifts: arXiv:0804.3263
- Universal partial fields: arXiv:0806.4487

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