Laten zien dat het niet past/ How to show it doesn't fit

Stefan van Zwam

Based on joint work with Rhiannon Hall and Dillon Mayhew

Nederlands Mathematisch Congres, April 15, 2009

TU/e Technische Universiteit Eindhoven University of Technology

Where innovation starts

Matroid theory

ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.[1]

By Hassler Whitney.

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1. Introduction. Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

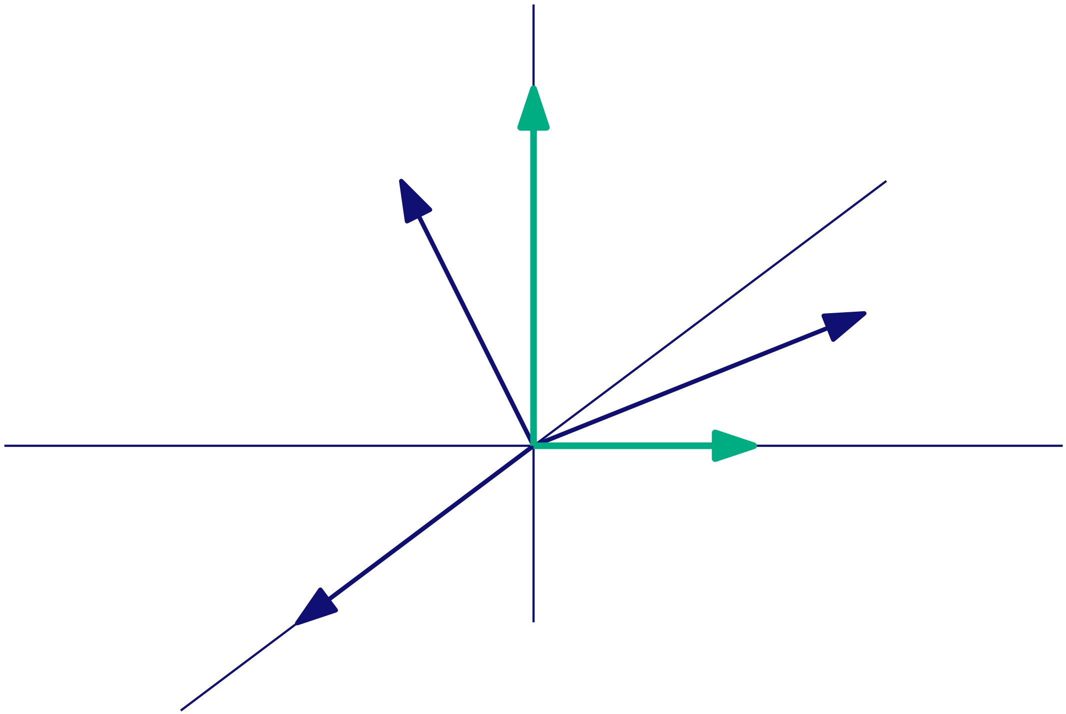
(a) Any subset of an independent set is independent.

(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.

Example

Linearly independent vectors in $\mathbb{R}^n$



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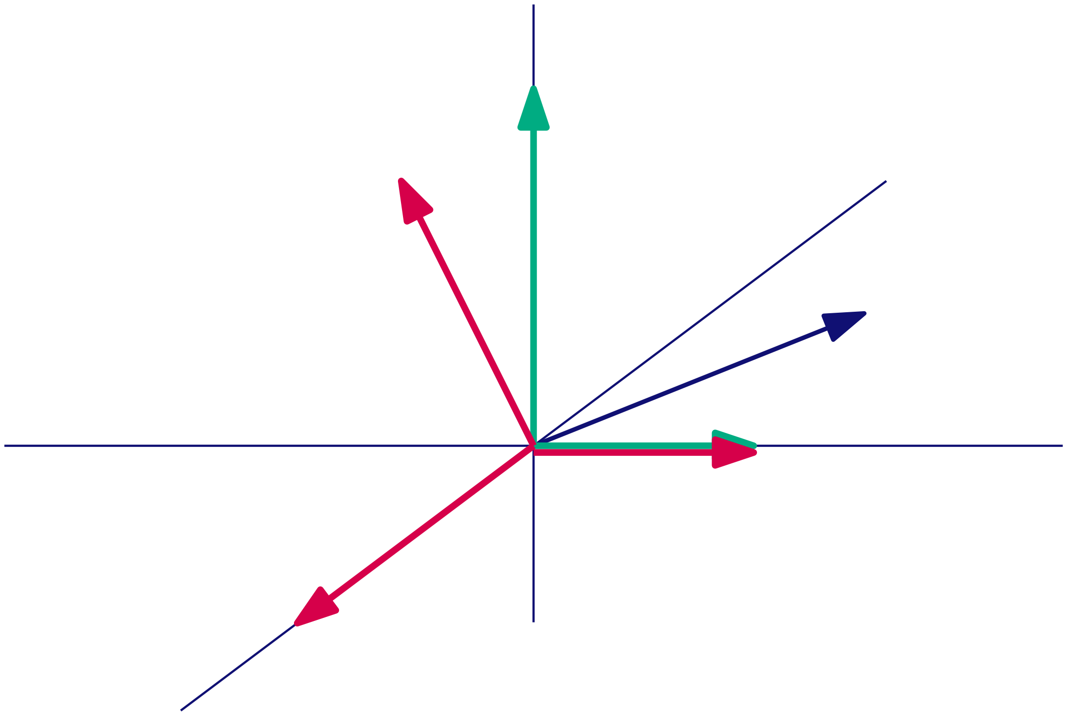
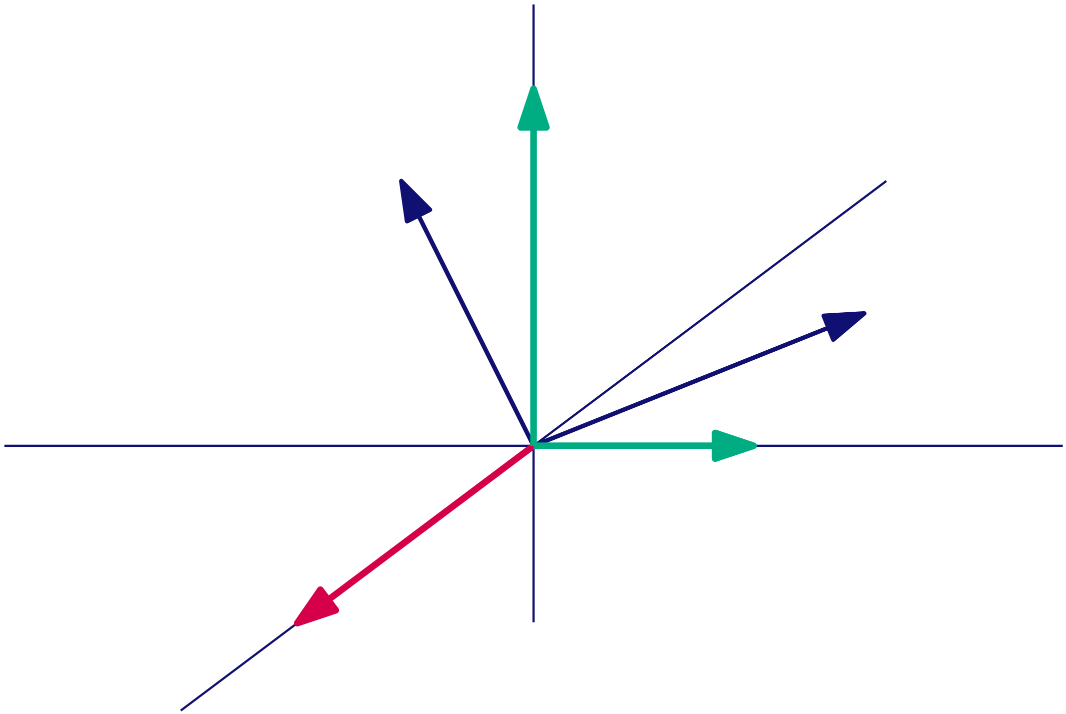


Linearly independent vectors in $\mathbb{R}^n$



Matroid axioms

Lemma. Given

E: finite set of vectors

$\mathcal{I}$: collection of linearly independent subsets

then

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

$$\exists e \in J \setminus I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

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Matroid axioms

Definition. Given

E: finite set

$\mathcal{I}$: collection of subsets

such that

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

$$\exists e \in J \setminus I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

Then $M = (E, \mathcal{I})$ is a **matroid**.

Matroids everywhere!

1899: Hilbert, Bernays: Plane geometry

1900–1936: Dedekind, Birkhoff, MacLane:
Semimodular lattices

1910–1937: Steinitz, Van der Waerden:
Algebraic dependence

1936: Nakasawa: Projective geometry

1935: Whitney: Lin. dependence, duality, graphs

1942: Rado: Transversals (matching theory)

1958: Tutte: Connectivity, minors, . . .

1971: Edmonds: Greedy algorithm

Rota, Brylawski, Seymour, . . .

Matroid representation

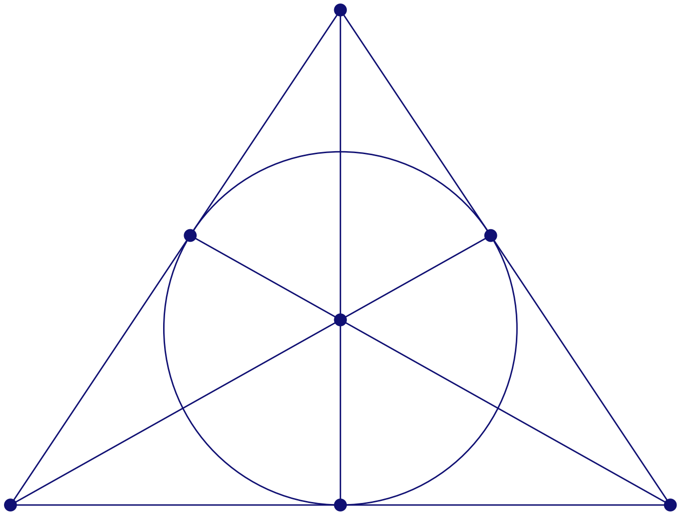
The representation problem

Problem. Is there a map

$$E \to \mathbb{F}^n$$

preserving the dependencies of $M = (E, \mathcal{I})$?

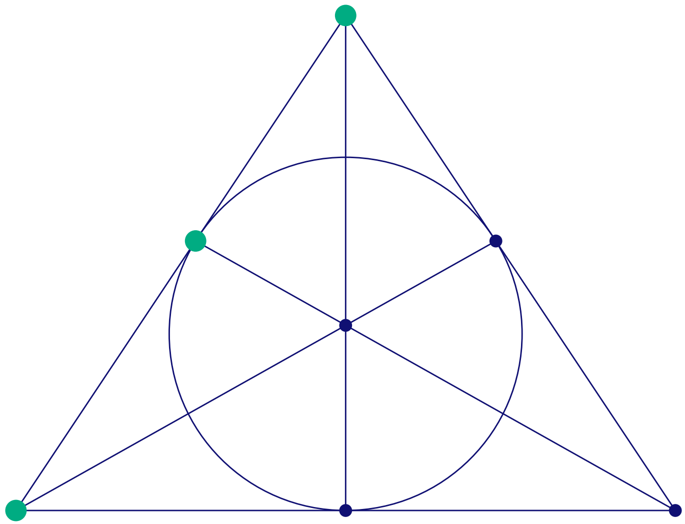
Example: the Fano matroid



- $E = \{$ points $\}$
- $\mathcal{I} = \{ X \subseteq E$ in general position $\}$

Matroid representation

Example: the Fano matroid



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Matroid representation

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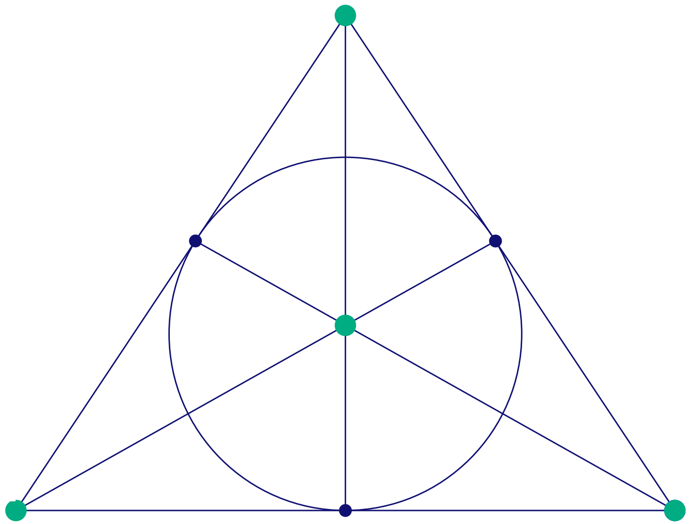

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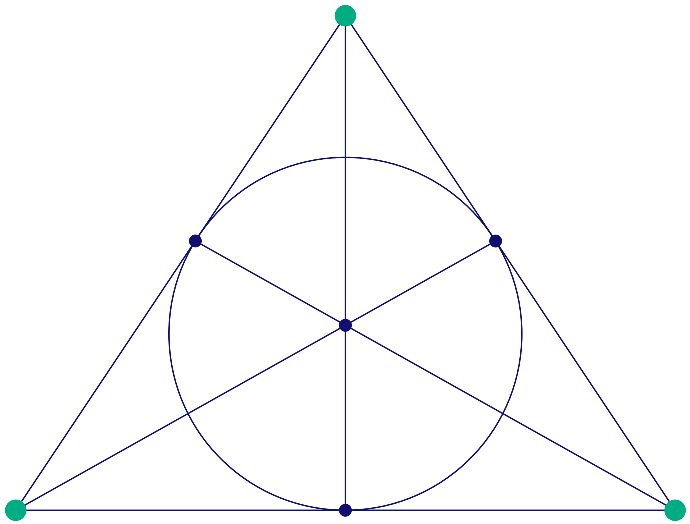

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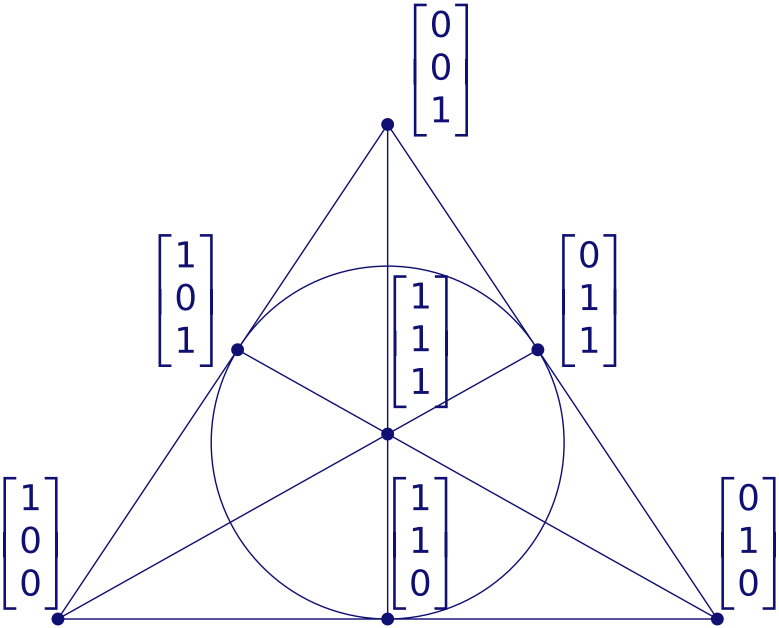

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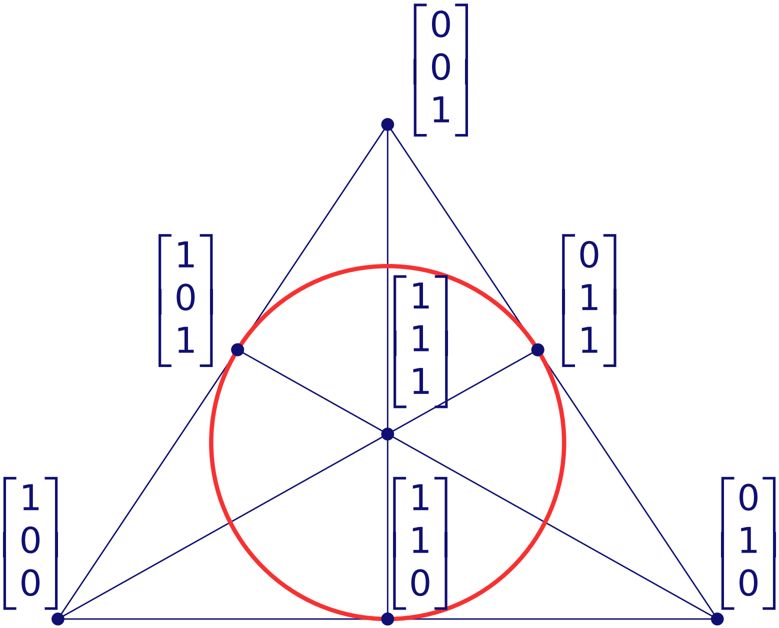

$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = - \begin{bmatrix}0\\0\\2\end{bmatrix}$$

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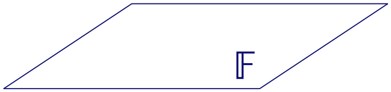
Matroid representation

How to show it doesn't fit?

Problem. Is there a dependency-preserving map

$E(M) \to$



?

Matroid representation

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- "Yes" certified by vectors $\{v_1, \ldots, v_n\}$

How to show it doesn't fit?

Problem. Is there a dependency-preserving map

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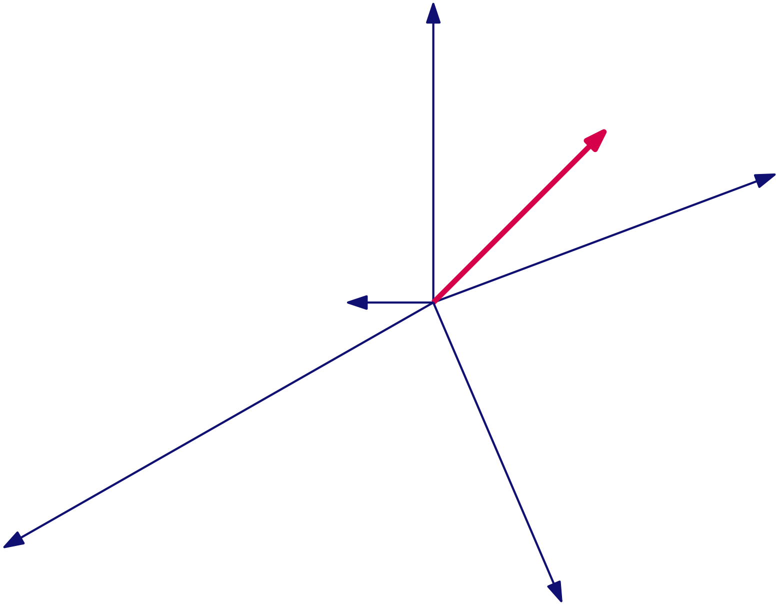
- "Yes" certified by vectors $\{v_1, \ldots, v_n\}$
- **How to certify "no"?**

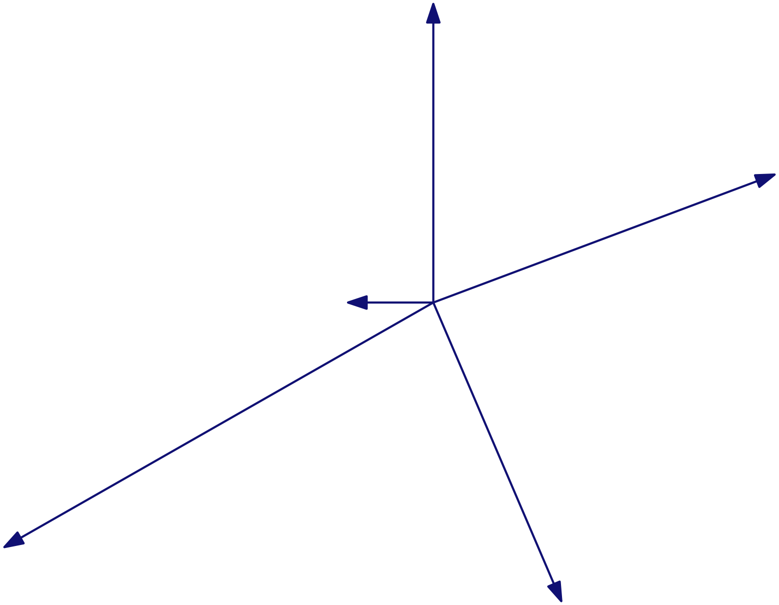
Matroid minors

Reducing a set of vectors: deletion



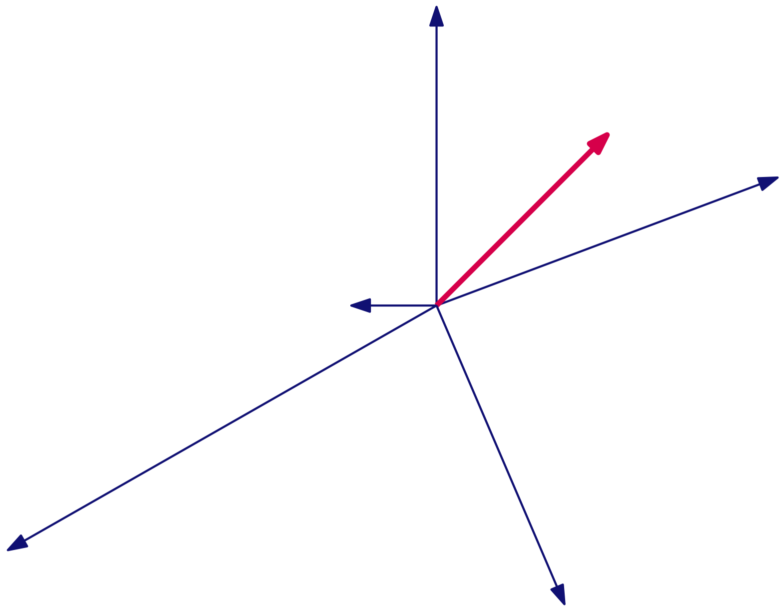
Matroid minors

Reducing a set of vectors: deletion



Matroid minors

Reducing a set of vectors: projection



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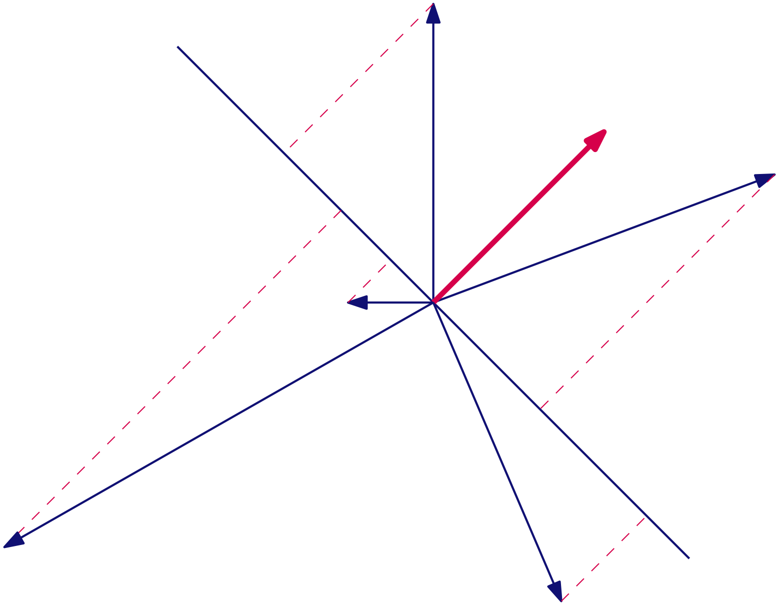

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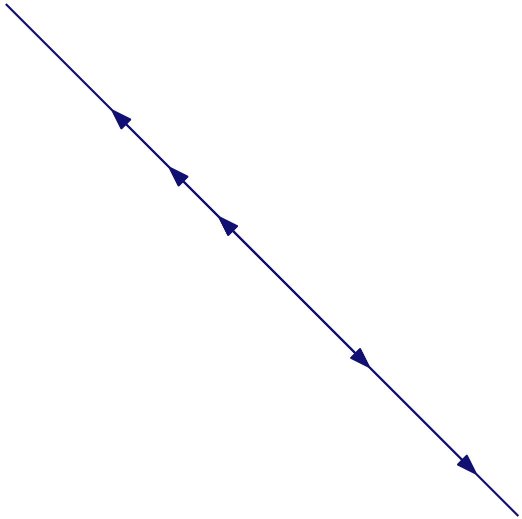

Matroid minors

Abstract definition

- *Deletion:* $M \backslash e := (E \setminus \{e\}, \{I \in \mathcal{I} : e \notin I\})$
- *Contraction:* $M/e := (E \setminus \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\})$
- *Minors:* Obtained from sequence of such steps

Matroid minors

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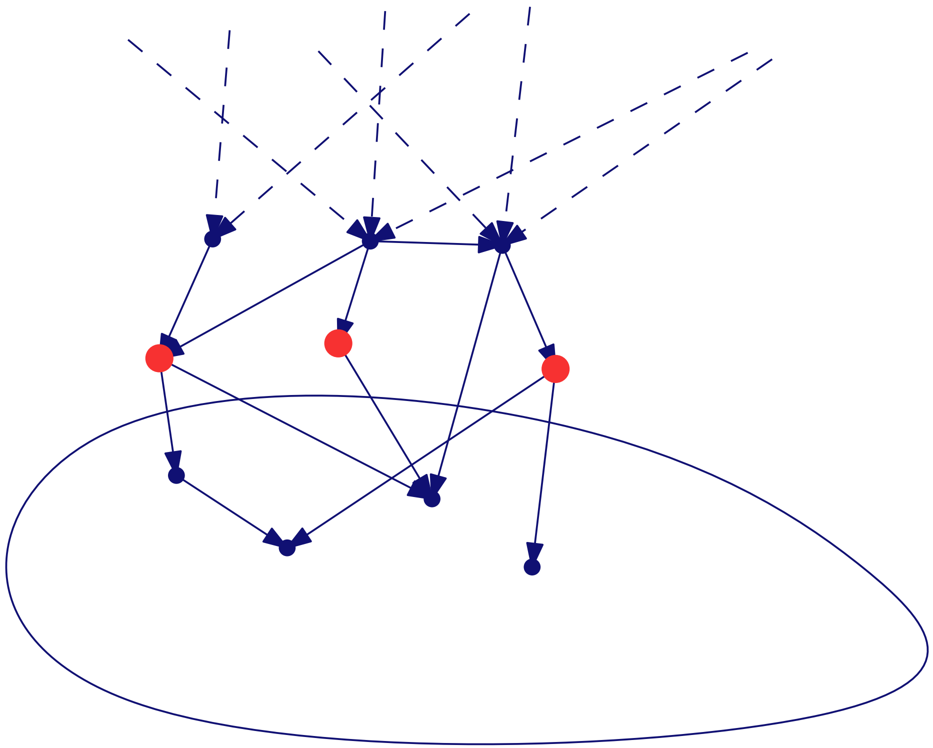
- *Minors:* Obtained from sequence of such steps
  - Generate partial order
  - Preserve representability

Excluded minors



Matroid representation

That's how!

Problem:
Is there a dependency-preserving map

$E(M) \rightarrow$ 

- **How to certify the answer is "no"?**
- By reducing to an excluded minor!

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Matroid representation

That's how!

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- **How to certify the answer is "no"?**
- By reducing to an excluded minor!
- Rota's Conjecture: finitely many

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Matroid representation

Rota’s Conjecture

Conjecture (Rota 1971): $\mathbb{F}$ finite, then $\exists k = k(\mathbb{F})$: exactly k excluded minors for

$$\left\{ M : E(M) \longrightarrow \right.$$

$$\left. \right\}$$

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- Proven for $\mathbb{F} \in \{GF(2), GF(3), GF(4)\}$

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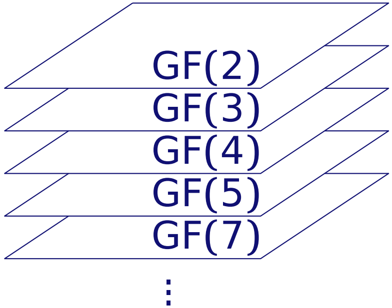
Regular matroids

Theorem (Tutte 1958):
Exactly 3 excluded minors for

$\Big\{ M : E(M) \to$

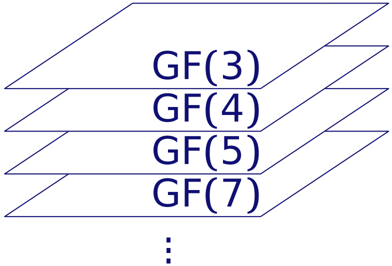


$\Big\}$

Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

$$\left\{ M : E(M) \longrightarrow \right.$$



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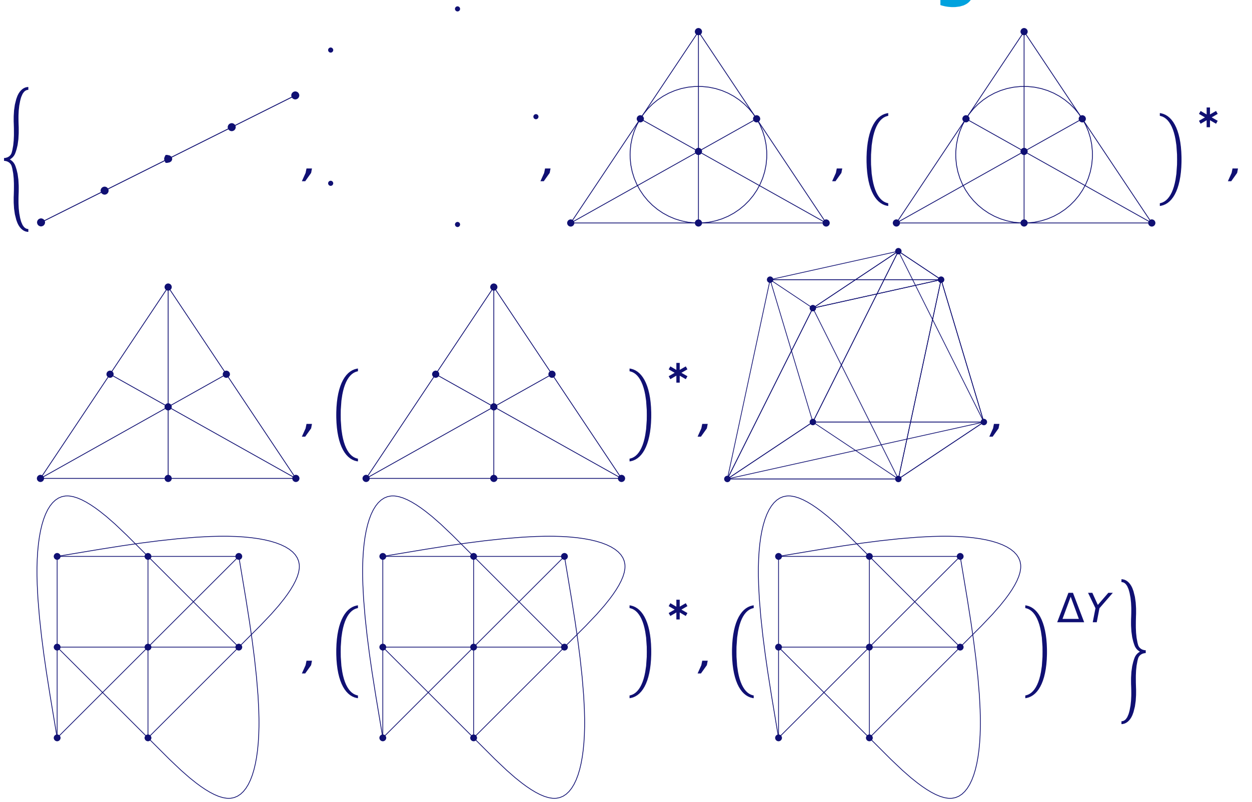
Excluded minors for near-regular



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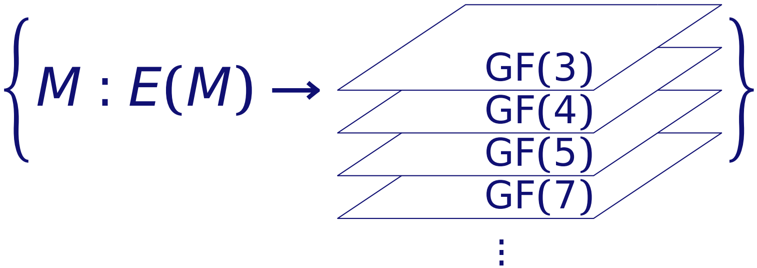

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Matroid representation

Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

$$\left\{ M : E(M) \to \mathbb{P} \right\}$$

Why care?

- Complexity of algorithms
- Structure of ternary matroids
- Sharpening the knives for Rota's Conjecture

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The end

Summary

- Matroid axioms abstract linear dependence
- Matroids occur everywhere
- Representation problem
- Excluded minors
- Near-regular matroids

Thank you for listening

Preprint at `http://www.win.tue.nl/~svzwam/`