Laten zien dat het niet past/
How to show it doesn’t fit

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Based on joint work with Rhiannon Hall and Dillon Mayhew

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ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.¹

By Hassler Whitney.

1. Introduction. Let $C_1, C_2, \ldots, C_n$ be the columns of a matrix $M$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

(a) Any subset of an independent set is independent.

(b) If $N_p$ and $N_{p+1}$ are independent sets of $p$ and $p+1$ columns respectively, then $N_p$ together with some column of $N_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in §16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.
Linearly independent vectors in $\mathbb{R}^n$
Example

Linearly independent vectors in $\mathbb{R}^n$
Example

Linearly independent vectors in $\mathbb{R}^n$
Lemma. Given

\( E \): finite set of vectors
\( \mathcal{I} \): collection of linearly independent subsets

then

- \( \emptyset \in \mathcal{I} \)
- \( J \in \mathcal{I} \) and \( I \subseteq J \), then \( I \in \mathcal{I} \)
- \( I, J \in \mathcal{I} \) and \( |I| < |J| \), then

\[ \exists e \in J \setminus I \text{ such that } I \cup \{e\} \in \mathcal{I} \]
Definition. Given

\( E: \) finite set

\( \mathcal{I}: \) collection of subsets

such that

- \( \emptyset \in \mathcal{I} \)
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\[ \exists e \in J \setminus I \text{ such that } I \cup \{e\} \in \mathcal{I} \]

Then \( M = (E, \mathcal{I}) \) is a matroid.
1899: Hilbert, Bernays: Plane geometry
1900–1936: Dedekind, Birkhoff, MacLane: Semimodular lattices
1910–1937: Steinitz, Van der Waerden: Algebraic dependence
1936: Nakasawa: Projective geometry
1935: Whitney: Lin. dependence, duality, graphs
1942: Rado: Transversals (matching theory)
1958: Tutte: Connectivity, minors, …
1971: Edmonds: Greedy algorithm
Rota, Brylawski, Seymour, …
The representation problem

Problem. Is there a map

\[ E \rightarrow \mathbb{F}^n \]

preserving the dependencies of \( M = (E, \mathcal{I}) \)?
Matroid representation

Example: the Fano matroid

- $E = \{ \text{points} \}$
- $\mathcal{I} = \{ X \subseteq E \text{ in general position} \}$
Matroid representation

Example: the Fano matroid

- $E = \{ \text{points} \}$
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Example: the Fano matroid

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- $I = \{ X \subseteq E \text{ in general position} \}$
Matroid representation

Example: the Fano matroid

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix} - \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 2 \\
\end{bmatrix}
\]
How to show it doesn’t fit?

**Problem.** Is there a dependency-preserving map

$$E(M) \rightarrow F$$
How to show it doesn’t fit?

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\[ E(M) \rightarrow F \]

- “Yes” certified by vectors \( \{v_1, \ldots, v_n\} \)
How to show it doesn’t fit?

**Problem.** Is there a dependency-preserving map

\[ E(M) \rightarrow \mathbb{F} \]

• “Yes” certified by vectors \( \{ v_1, \ldots, v_n \} \)

• **How to certify “no”**?
Reducing a set of vectors: deletion
Reducing a set of vectors: deletion
Reducing a set of vectors: projection
Reducing a set of vectors: projection
Reducing a set of vectors: projection
Abstract definition

• **Deletion**: \( M \setminus e := (E \setminus \{e\}, \{I \in \mathcal{I} : e \notin I\}) \)

• **Contraction**: \( M/e := (E \setminus \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\}) \)

• **Minors**: Obtained from sequence of such steps
Abstract definition

- **Deletion**: $M \setminus e := (E \setminus \{e\}, \{I \in \mathcal{I} : e \notin I\})$
- **Contraction**: $M/e := (E \setminus \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\})$
- **Minors**: Obtained from sequence of such steps
  - Generate partial order
  - Preserve representability
Matroid minors

Excluded minors
That’s how!

**Problem:**
Is there a dependency-preserving map

\[ E(M) \rightarrow GF(2) \]

- How to certify the answer is “no”?
- By reducing to an excluded minor!
That’s how!

Problem:
Is there a dependency-preserving map

\[ E(M) \rightarrow GF(2) \]

- How to certify the answer is “no”?
- By reducing to an excluded minor!
- Rota’s Conjecture: finitely many
Rota’s Conjecture

Conjecture (Rota 1971): If \( F \) is finite, then there exists \( k = k(F) \): exactly \( k \) excluded minors for \( M : E(M) \rightarrow \mathbb{F} \)
Rota’s Conjecture

Conjecture (Rota 1971): \( F \) finite, then \( \exists k = k(F) : \) exactly \( k \) excluded minors for

\[
\left\{ M : E(M) \rightarrow GF(F) \right\}
\]

- Proven for \( F \in \{ GF(2), GF(3), GF(4) \} \)
Regular matroids

**Theorem (Tutte 1958):**
Exactly 3 excluded minors for

\[
\left\{ M : E(M) \rightarrow \begin{align*}
&\text{GF}(2) \\
&\text{GF}(3) \\
&\text{GF}(4) \\
&\text{GF}(5) \\
&\text{GF}(7) \\
&\vdots
\end{align*} \right\}
\]
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[ \{ M : E(M) \rightarrow \{ \text{GF(3)}, \text{GF(4)}, \text{GF(5)}, \text{GF(7)}, \ldots \} \} \]
Excluded minors for near-regular

\[
\{ \text{\begin{itemize}
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\end{itemize}} \} \}
\]
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009): Exactly 10 excluded minors for

\[
\left\{ M : E(M) \rightarrow \begin{array}{c} GF(3) \\ GF(4) \\ GF(5) \\ GF(7) \\ \vdots \end{array} \right\}
\]
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[
\left\{ M : E(M) \rightarrow P \right\}
\]

Matroid representation
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[ M : E(M) \rightarrow \mathbb{GF}(2) \]

Why care?
- Complexity of algorithms
- Structure of ternary matroids
- Sharpening the knives for Rota’s Conjecture
Summary

- Matroid axioms abstract linear dependence
- Matroids occur everywhere
- Representation problem
- Excluded minors
- Near-regular matroids

Thank you for listening

Preprint at http://www.win.tue.nl/~svzwam/