Partial Fields and Rota's Conjecture

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Based on joint work with Rhiannon Hall, Dillon Mayhew, Rudi Pendavingh, and Geoff Whittle

Technische Universiteit
Eindhoven
University of Technology

Seminar Incidence Geometry, Gent, Belgium, June 12, 2009

Overview

I. Matroids, representations, Rota's Conjecture

II. Partial fields

III. Partial fields and Rota's Conjecture

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Matroids, representations, Rota’s Conjecture

Matroid theory

ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.[1]

By Hassler Whitney.

1. Introduction. Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

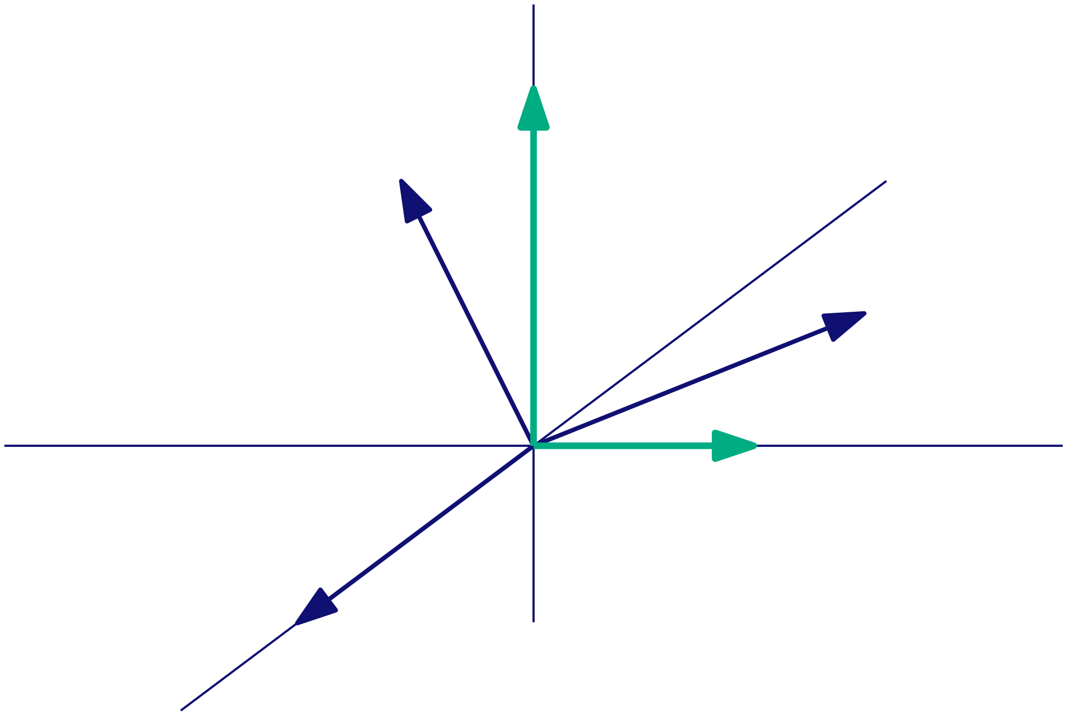
(a) Any subset of an independent set is independent.

(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.

Example

Linearly independent vectors in $\mathbb{R}^n$



Example

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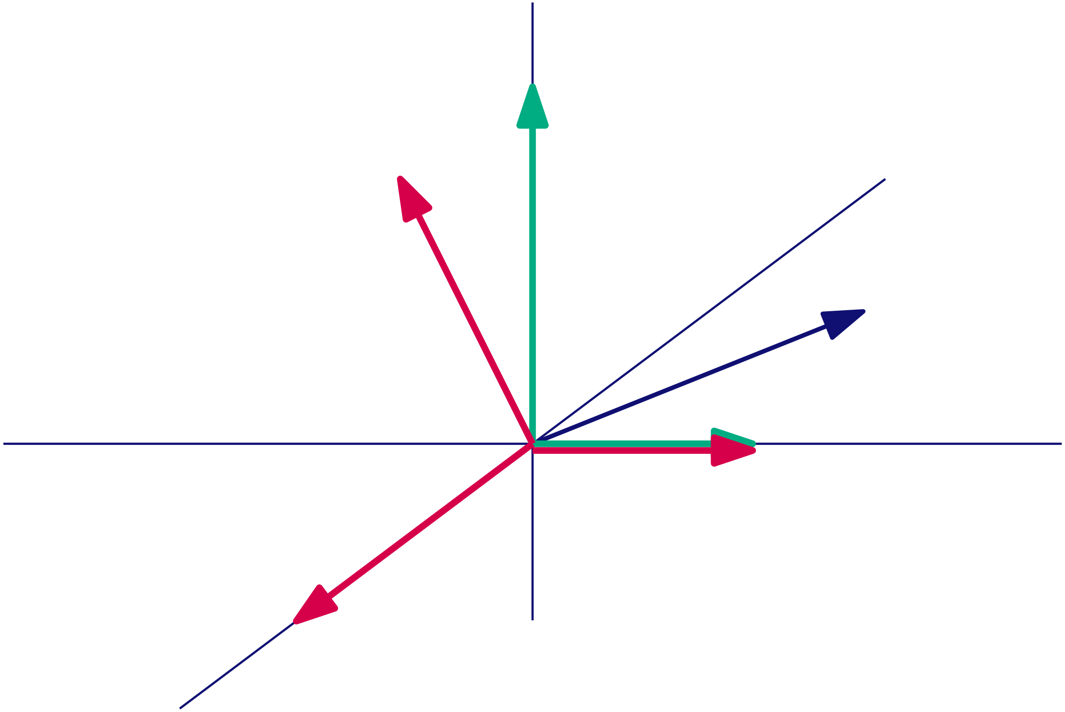
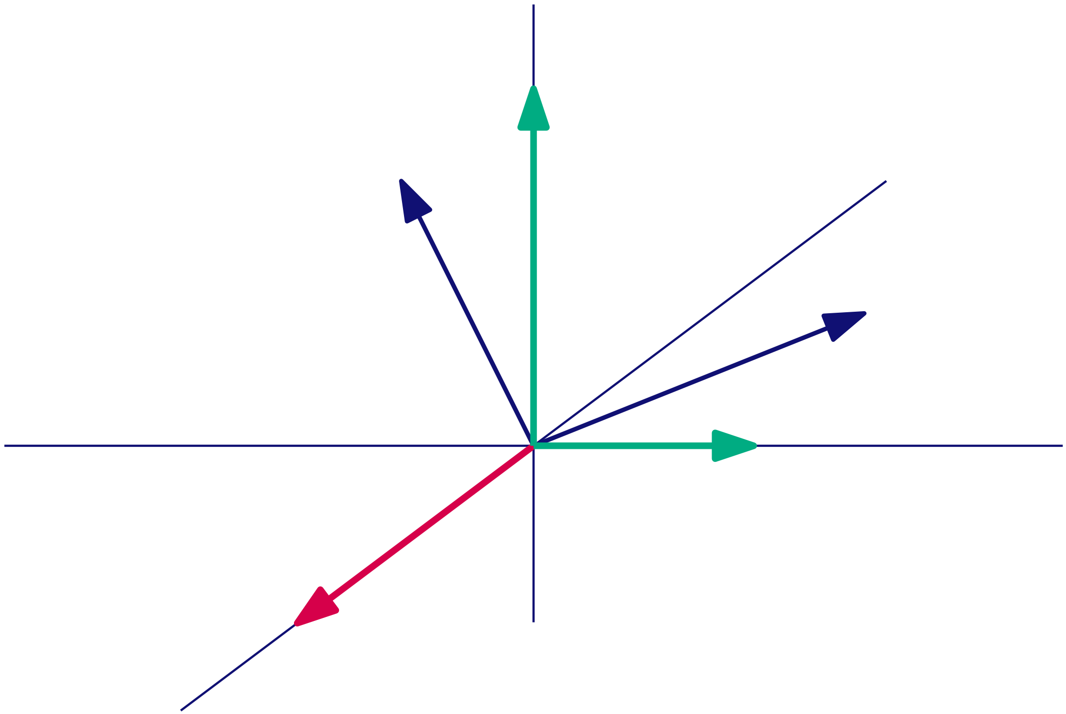


Example

Linearly independent vectors in $\mathbb{R}^n$



Matroid axioms

Lemma. Given

E: finite set of vectors

$\mathcal{I}$: collection of linearly independent subsets

then

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

$$\exists e \in J \setminus I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

Matroid axioms

Definition. Given

E: finite set

$\mathcal{I}$: collection of subsets

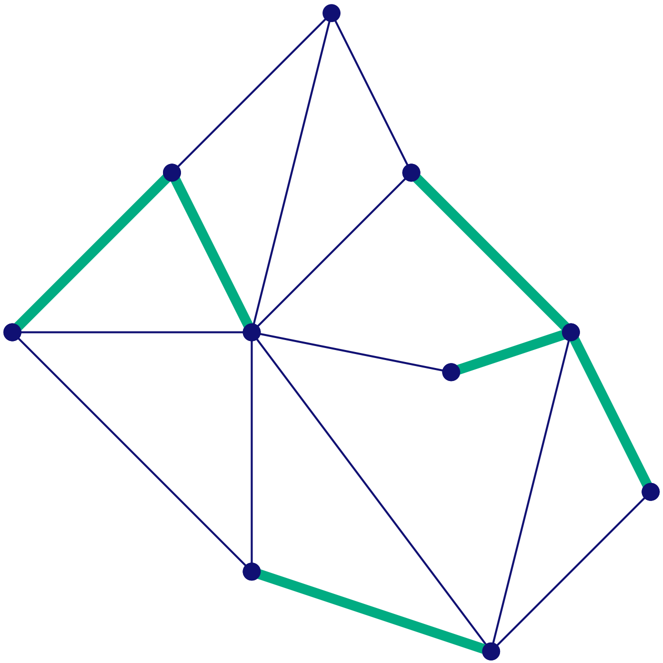
such that

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

$$\exists e \in J \setminus I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

Then $M = (E, \mathcal{I})$ is a **matroid**.

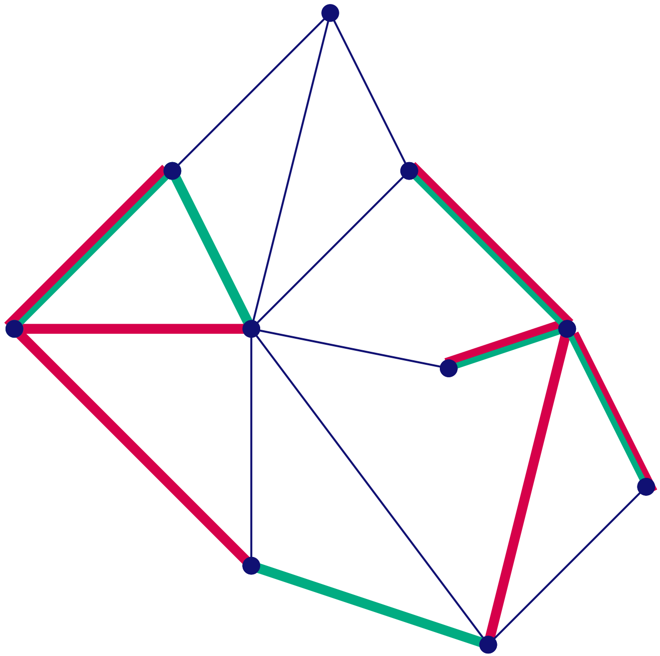
Forests in a graph



Forest I.

Example

Forests in a graph



$\exists e \in J \setminus I$ such that $I \cup \{e\}$ forest.

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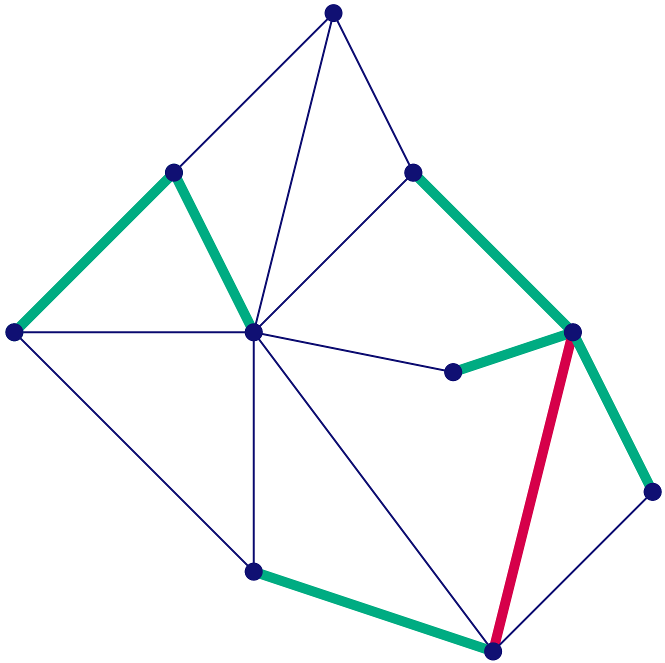
Example

Forests in a graph



Forest $I \cup \{e\}$.

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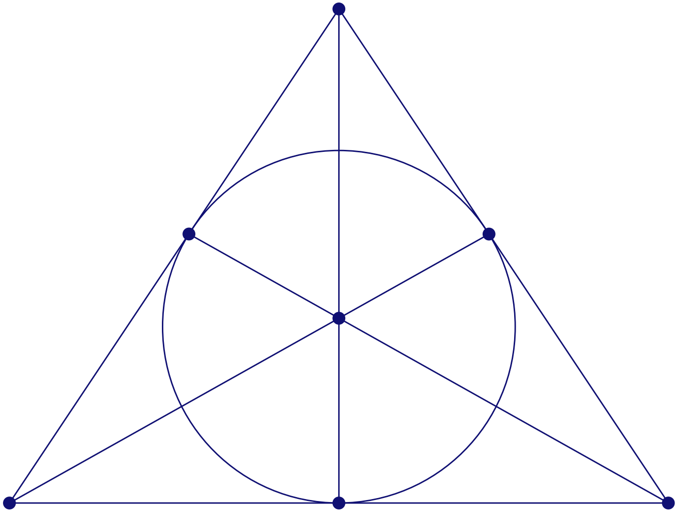
Incidence geometry!

Example: the Fano matroid



- $E = \{$ points $\}$
- $\mathcal{I} = \{ X \subseteq E$ in general position $\}$

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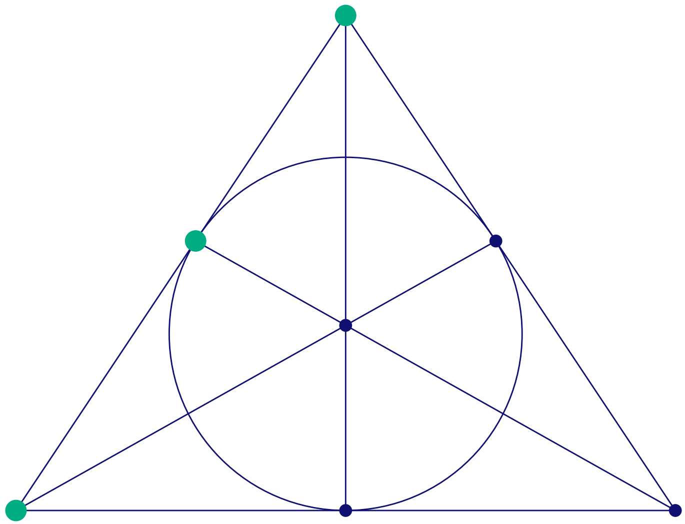
Incidence geometry!

Example: the Fano matroid



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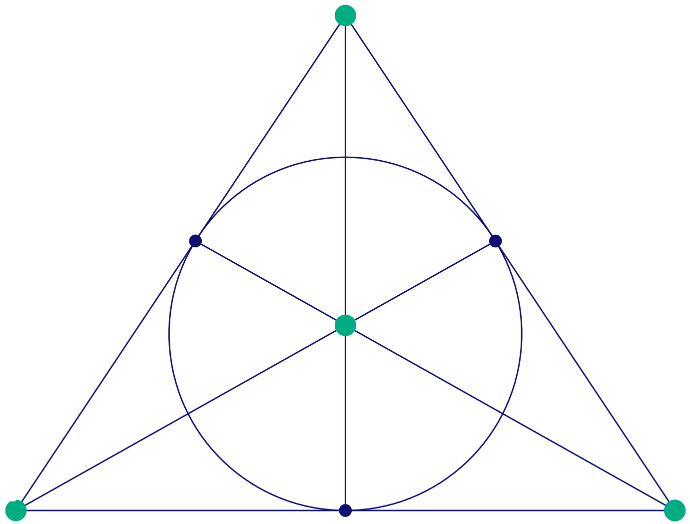
Incidence geometry!

Example: the Fano matroid



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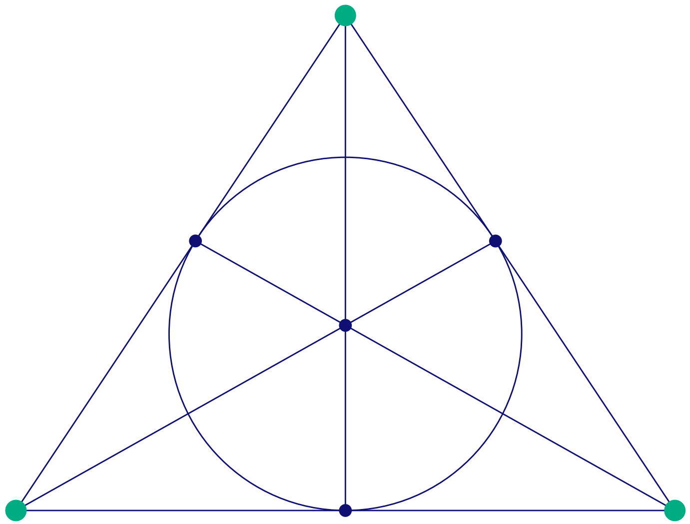
Example: the Fano matroid



- $E = \{ \text{points} \}$
- $\mathcal{I} = \{ X \subseteq E \text{ in general position } \}$

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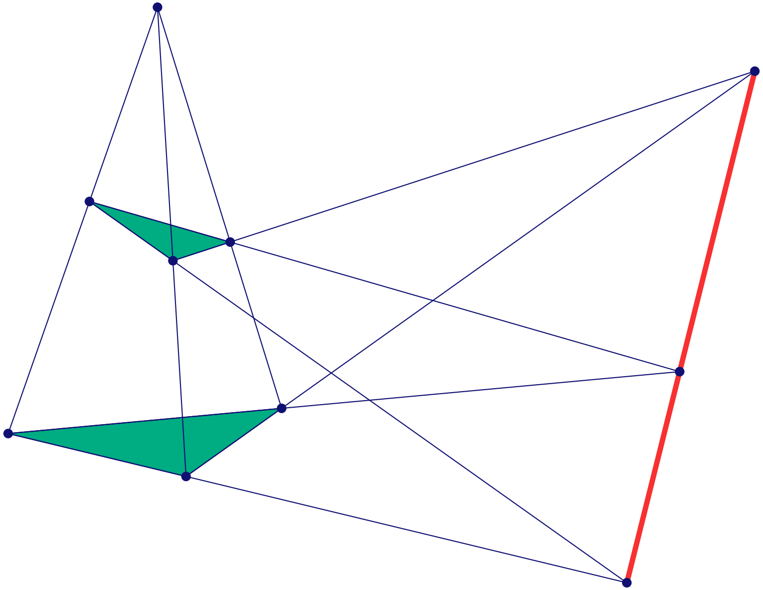
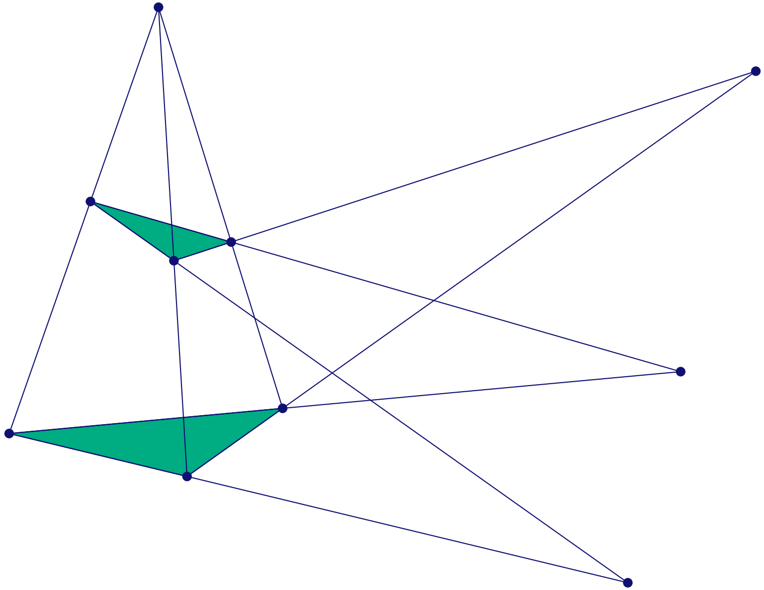
Incidence geometry?

Strange example: the Non-Desargues matroid



Strange example: the Non-Desargues matroid



Matroid representation

The representation problem

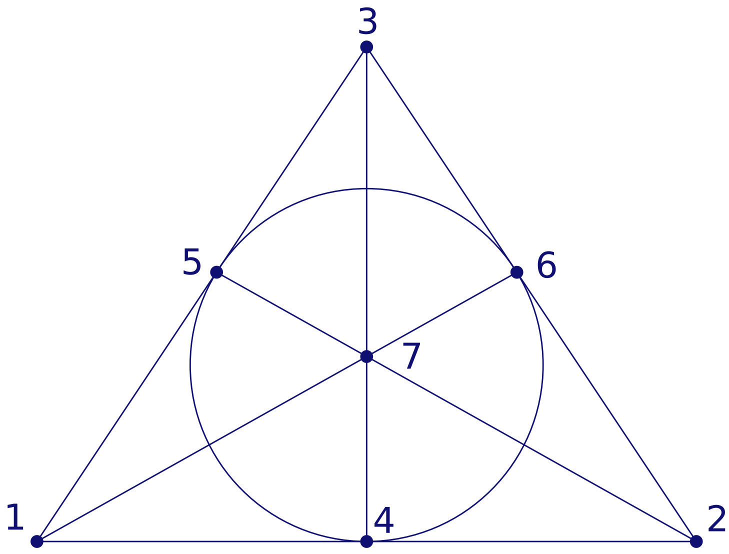
Problem. Is there a map

$$E \rightarrow \mathbb{F}^n$$

preserving the dependencies of $M = (E, \mathcal{I})$?

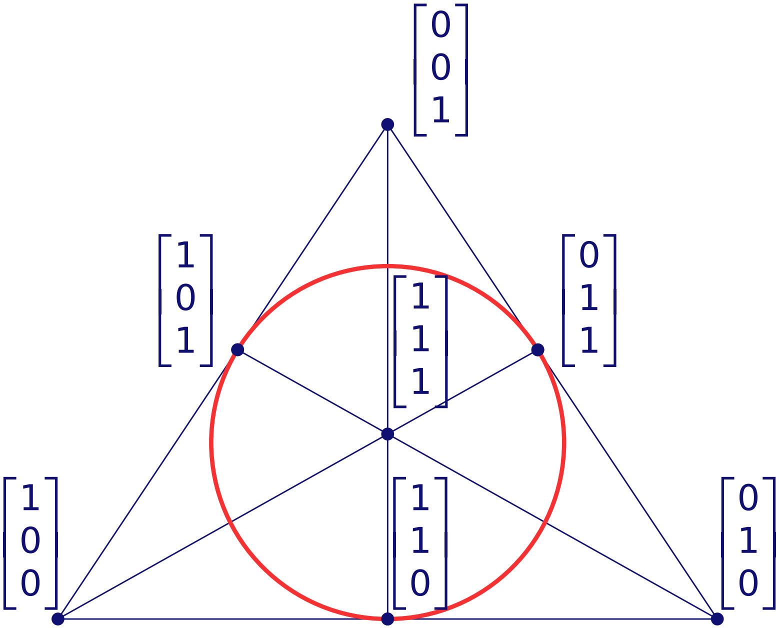
Matroid representation

Example: the Fano matroid



- $E = \{$ points $\}$
- $\mathcal{I} = \{ X \subseteq E$ in general position $\}$

Example: the Fano matroid



$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = -\begin{bmatrix}0\\0\\2\end{bmatrix}$$

Matroid representation

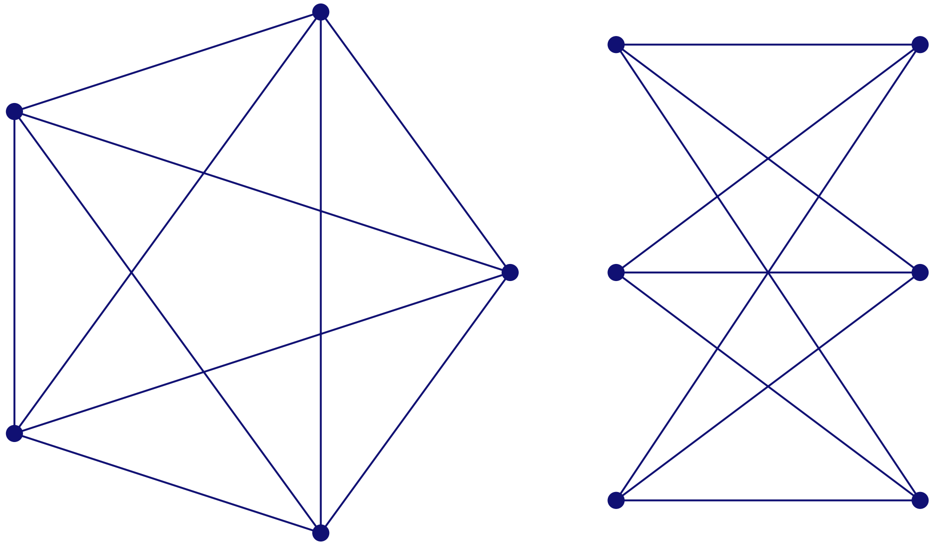
Problem. Is there a dependency-preserving map

$E(M) \longrightarrow$  ?

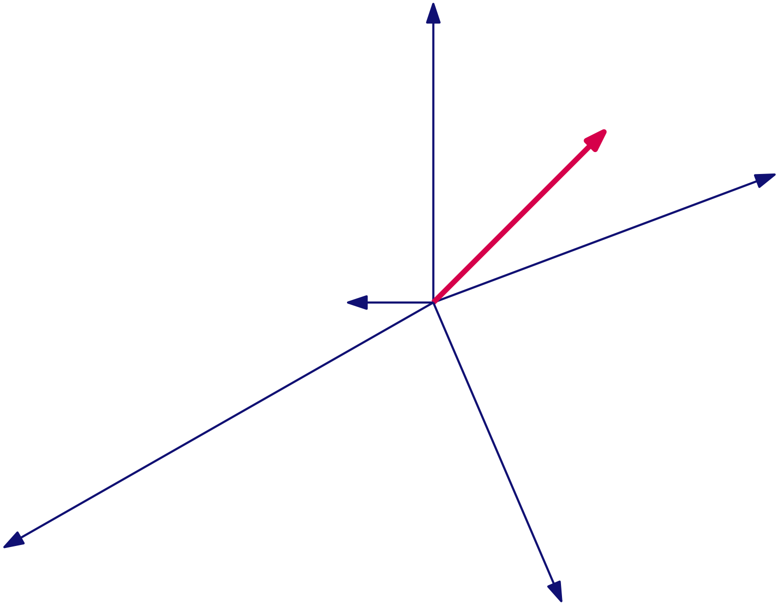
- "Yes" certified by vectors $\{v_1, \ldots, v_n\}$
- **How to certify "no"?**

Graph minors

Theorem (Kuratowski):
Graph is planar ⇔ no minor isomorphic to



Reducing a set of vectors: deletion



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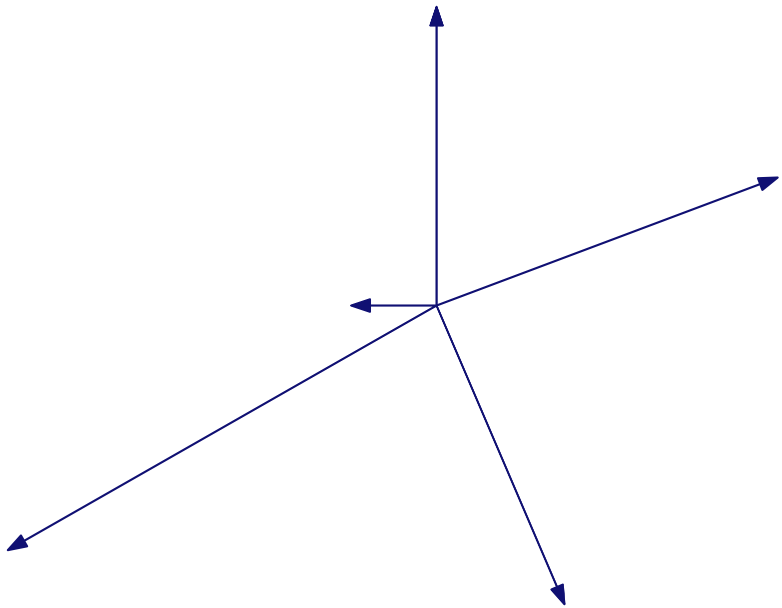
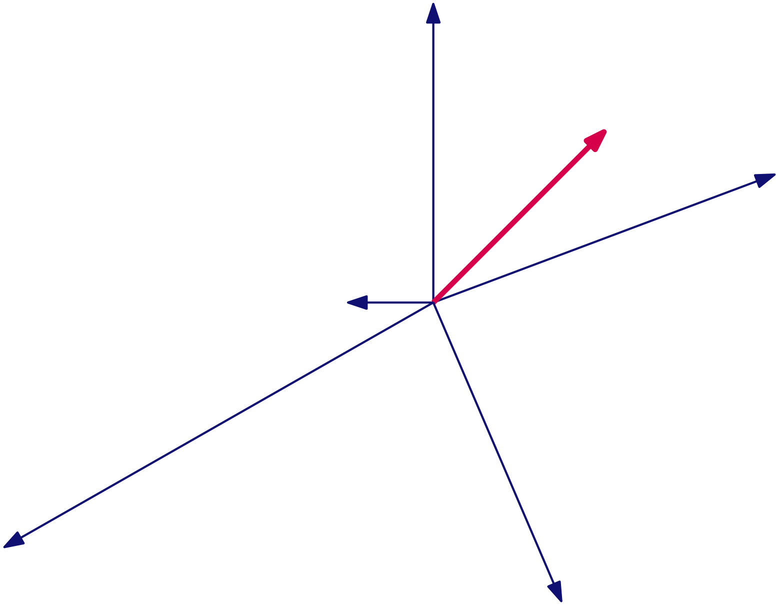


Reducing a set of vectors: projection



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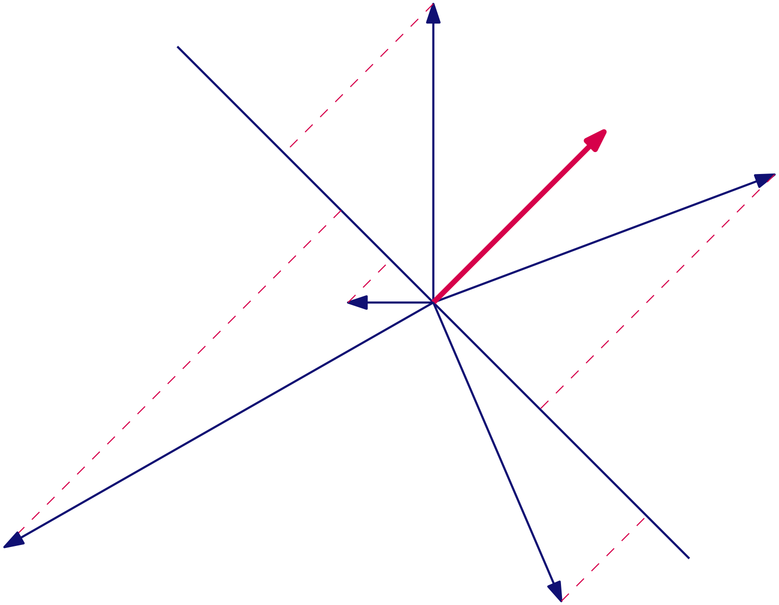
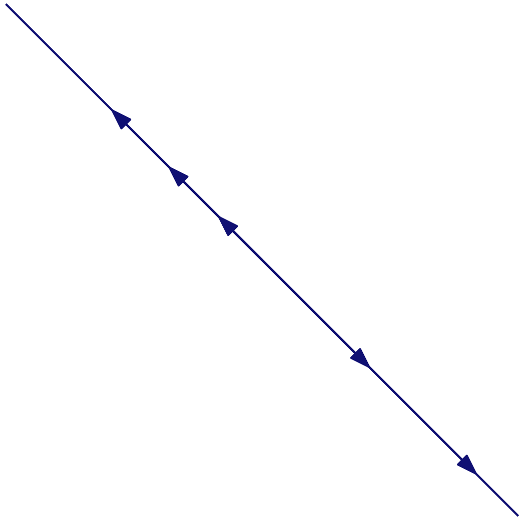


Matroid minors

Reducing a set of vectors: projection



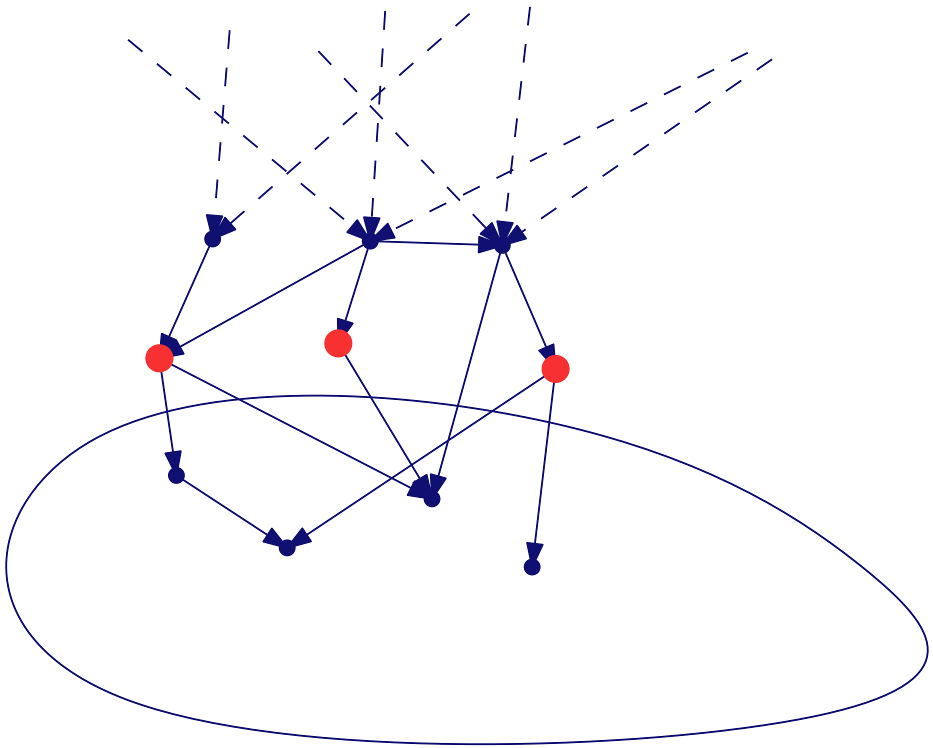
Matroid minors

Abstract definition

- *Deletion:* $M \backslash e := (E \setminus \{e\}, \{I \in \mathcal{I} : e \notin I\})$
- *Contraction:* $M/e := (E \setminus \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\})$
- *Minors:* Obtained from sequence of such steps
  - Generate partial order
  - Preserve representability

Matroid minors

Excluded minors



Matroid representation

Problem:

Is there a dependency-preserving map

$E(M) \longrightarrow$

- **How to certify the answer is “no”?**
- By reducing to an excluded minor!
- Rota’s Conjecture: finitely many

Matroid representation

Rota’s Conjecture

Conjecture (Rota 1971): $\mathbb{F}$ finite, then $\exists k = k(\mathbb{F})$: exactly k excluded minors for

$$\left\{ M : E(M) \rightarrow \right.$$

$$\left. \right\}$$

- Proven for $\mathbb{F} \in \{\mathrm{GF}(2), \mathrm{GF}(3), \mathrm{GF}(4)\}$

Matroid representation

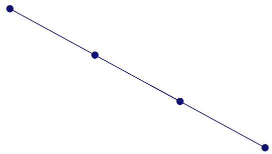
Rota's Conjecture

Theorem (Tutte 1958):
Exactly 1 excluded minor for

$$\left\{ M : E(M) \longrightarrow \right.$$

$$\left. \right\}$$

namely



Rota's Conjecture

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$$\left. \right\}$$

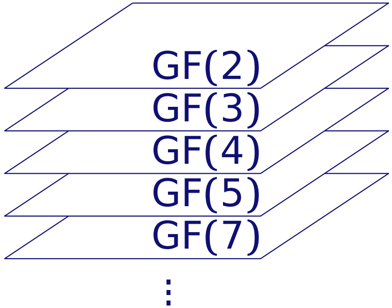
$\mathbb{F}$	GF(2)	GF(3)	GF(4)	GF(5)
k	1	3	7	≥ 564[a]

[a] Mayhew, Royle 2009

Regular matroids

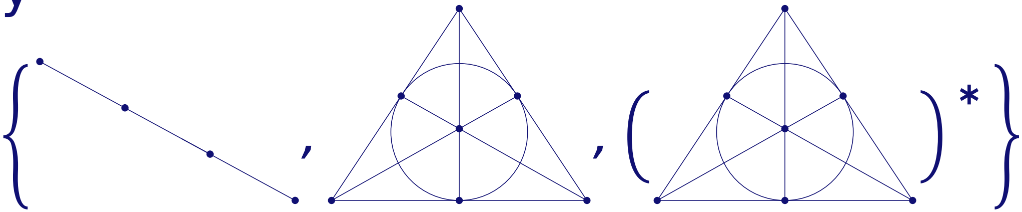
Theorem (Tutte 1958):
Exactly 3 excluded minors for

$\left\{ M : E(M) \rightarrow \right.$



$\left. \right\}$

namely

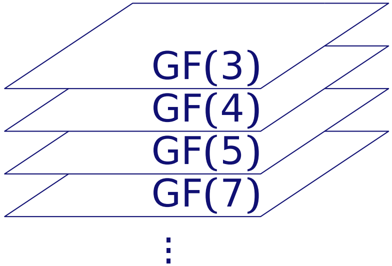


Matroid representation

Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

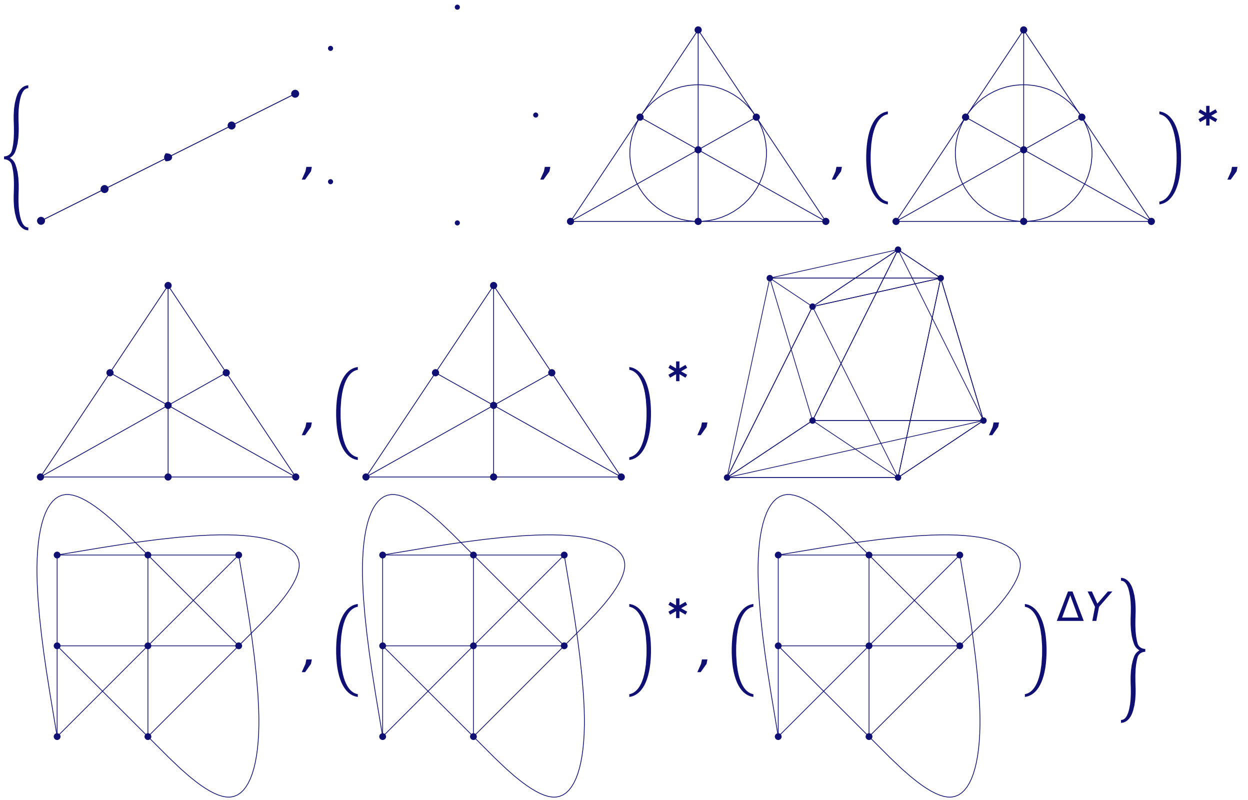
$$\left\{ M : E(M) \to \right.$$



$$\left. \right\}$$

Matroid representation

namely



Matroid representation

Near-regular matroids

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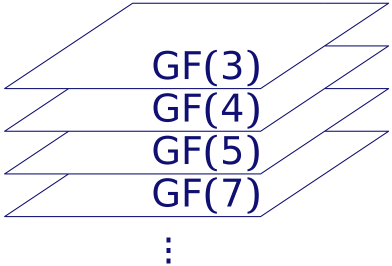

$$\left. \right\}$$

Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
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$$\left. \right\}$$

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Partial Fields

Total unimodularity

Theorem (Tutte 1958):
Equivalent for matroid M:

- M regular
- M has totally unimodular representation over $\mathbb{R}$

Definition:
Matrix A is totally unimodular $\Leftrightarrow$ every subdeterminant is in $\{-1, 0, 1\}$

Partial fields

Definition: Partial field $\mathbb{P} := (R, G)$

- R commutative ring
- $G \subseteq R^*$ group
- $-1 \in G$

Definition: Weak $\mathbb{P}$-matrix:

- $r \times E$ matrix A over R
- $\det(B) \in G \cup 0 \ \forall \ r \times r$ submatrix B

Theorem (vZ, Pendavingh 2009)

$$\{B \subseteq E \mid |B| = r, \det(A[r, B]) \neq 0\}$$

is set of bases of matroid M.
(Strengthening of [Semple, Whittle 1996])

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Partial fields

Strong $\mathbb{P}$-matrix

- Definition: *Every subdeterminant in* $G \cup 0$
- Example: weak $\mathbb{P}$-matrix of form $[I\,A]$
- $\Rightarrow$ every weak $\mathbb{P}$-matrix equivalent to strong!

Example:

$$\mathbb{U}_0 = (\mathbb{Z}, \{-1, 1\})$$

Strong $\mathbb{U}_0$-matrix is *totally unimodular*

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Partial fields

Homomorphisms

Definition: $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$ is *homomorphism* if, for $p, q \in G_1 \cup 0$,

- $\varphi(1) = 1$
- $\varphi(p)\varphi(q) = \varphi(pq) \in G_2 \cup 0$
- If $p + q \in G \cup 0$ then $\varphi(p) + \varphi(q) = \varphi(p+q) \in G_2 \cup 0$

Theorem (Semple, Whittle 1996):
A is strong $\mathbb{P}_1$-matrix
$\Rightarrow$
$\varphi(A)$ is strong $\mathbb{P}_2$-matrix

Also, $\det(A[X, Y]) = 0 \Leftrightarrow \det(\varphi(A)[X, Y]) = 0$

Product partial field

$$\mathbb{P}_1 \times \mathbb{P}_2 := (R_1 \times R_2, G_1 \times G_2)$$

Theorem (Pendavingh, vZ 2009):
Matroid representable over both $\mathbb{P}_1$ and $\mathbb{P}_2$
$\Longleftrightarrow$
Matroid representable over $\mathbb{P}_1 \times \mathbb{P}_2$

Partial fields

Regular matroids

Theorem (Tutte 1958):

Equivalent for matroid M:

(i) M representable over GF(2) and GF(3)

(ii) M representable over $\mathbb{U}_0 = (\mathbb{Z}, \{-1, 1\})$

(iii) M representable over all fields

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Partial fields

Dyadic matroids

Theorem (Whittle 1995):

Equivalent for matroid M:

(i) M representable over GF(3) and GF(5)

(ii) M representable over $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle)$

(iii) M representable over $\mathbb{F}$ unless $\chi(\mathbb{F}) = 2$

Golden ratio matroids

Theorem (Vertigan (unpublished) Pendavingh, vZ 2009):

Equivalent for matroid M:

(i) M representable over GF(4) and GF(5)

(ii) M representable over $\mathbb{U}_0 = (\mathbb{R}, \langle -1, \tau \rangle)$

(iii) M representable over GF(p) when $p \equiv \pm 1 \mod 5$

τ is *golden ratio*, root of $x^2 - x - 1 = 0$

Near-regular matroids

Theorem (Whittle 1997):
Equivalent for matroid M:

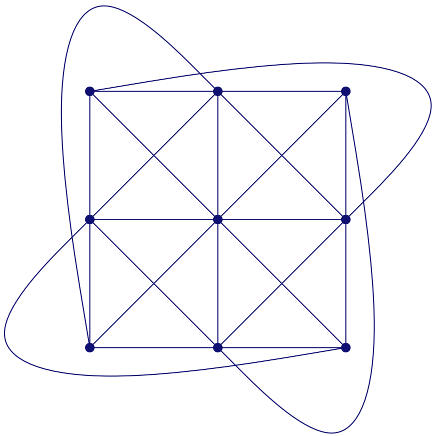
(i) M representable over GF(3) and GF(4) and GF(5)

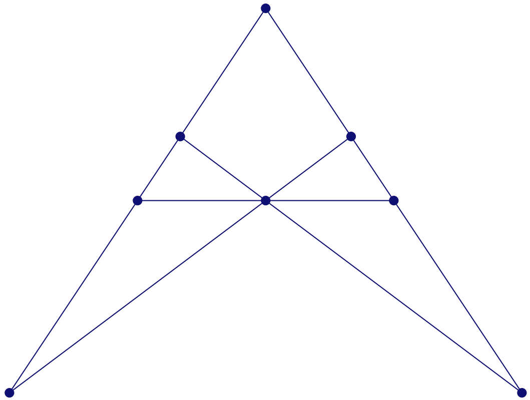
(ii) M representable over $\mathbb{U}_0 = (\mathbb{Q}(\alpha), \langle -1, \alpha, 1-\alpha \rangle)$

(iii) M representable over all fields with ≥ 3 elements

Hard implication is $(i) \Rightarrow (ii)$
Use *Lift Theorem* (Pendavingh, vZ 2009)

No universal model





Maximum-sized rank-3 in $\sqrt[6]{1}$ partial field (Oxley, Vertigan, Whittle 1998)

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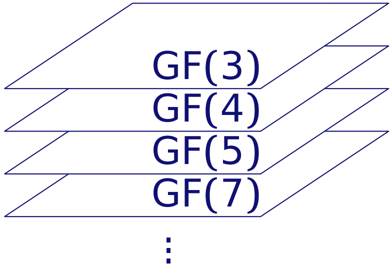
Partial Fields and Rota’s Conjecture

Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

$$\left\{ M : E(M) \longrightarrow \right.$$



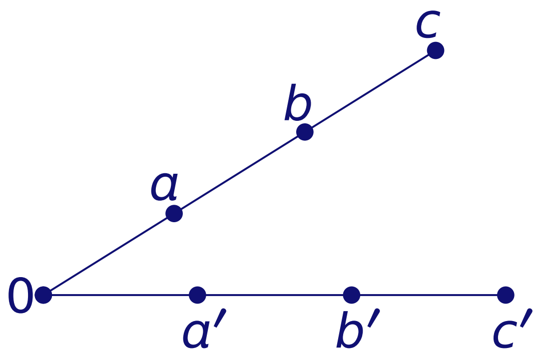
$$\left. \right\}$$

Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

$$\left\{ M : E(M) \to \mathbb{U}_1 \right\}$$

Non-unique representability



Rota's Conjecture

Recovering uniqueness

Connectivity!

- Splitter Theorem (Seymour 1981)
- Stabilizer Theorem (Whittle 1996)
- Blocking Sequences (Geelen et al. 2000)
- Branch Width (Geelen and Whittle 2002; Mayhew, Whittle, vZ 2009)
- ...

$\longrightarrow$ strategy for GF(5)

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Quinary matroids

Theorem (Pendavingh, vZ 2009):
M 3-connected matroid.

- At least two inequivalent representations over GF(5) $\Rightarrow$ representable over GF(p) when $p \equiv 1 \mod 4$
- At least three inequivalent representations over GF(5) $\Rightarrow$ representable over $\mathbb{F}$ if $|\mathbb{F}| \geq 5$
- At least five inequivalent representations over GF(5) $\Rightarrow$ six inequivalent representations

From partial fields

$$\mathbb{H}_6 = \mathbb{H}_5 \to \mathbb{H}_4 \to \mathbb{H}_3 \to \mathbb{H}_2 \to \mathbb{H}_1 = \mathrm{GF}(5)$$

That's all, folks!

Thank you for listening.

Preprints, slides at
http://www.win.tue.nl/~svzwam/