Representing some non-representable matroids

Stefan van Zwam

Based on joint work with Rudi Pendavingh

AMS Sectional Meeting Lexington, Kentucky, March 27, 2010
Overview

I. Partial fields: a crash course
II. Skew partial fields
III. A matroid
Part I
Partial Fields: a crash course
Total unimodularity

Definition:
Weakly unimodular matrix:

- $r \times E$ matrix $A$ over $\mathbb{Z}$
- $\det(D) \in \{-1, 1\} \cup 0$, $\forall$ $r \times r$ submatrix $D$, not all 0

\[
\begin{bmatrix}
e & f & g \\
1 & 1 & 0 & 1 \\
2 & 3 & -1 & 2
\end{bmatrix}
\]
Total unimodularity

Definition:
Weakly unimodular matrix:

- $r \times E$ matrix $A$ over $\mathbb{Z}$
- $\det(D) \in \{−1, 1\} \cup 0$, $\forall r \times r$ submatrix $D$, not all 0

Theorem (Tutte 1958):
Equivalent for matroid $M$:

- $M$ regular
- $M$ has weakly unimodular representation
Partial fields

Definition:

- \( r \times E \) matrix \( A \) over \( R \)
- \( \text{det}(D) \in G \cup 0 \), for all \( r \times r \) submatrix \( D \), not all 0
Partial fields

Definition:
(Weak) $\mathbb{P}$-matrix:

- $r \times E$ matrix $A$ over $R$
- $\det(D) \in G \cup 0$, $\forall$ $r \times r$ submatrix $D$, not all 0

Definition:
Partial field $\mathbb{P} := (R, G)$

- $R$ commutative ring
- $G \subseteq R^*$ group
- $-1 \in G$
Partial fields

Definition:
(Weak) $\mathcal{P}$-matrix:
- $r \times E$ matrix $A$ over $R$
- $\det(D) \in G \cup 0$, $\forall$ $r \times r$ submatrix $D$, not all 0

Definition:
Partial field $\mathcal{P} := (R, G)$
- $R$ commutative ring
- $G \subseteq R^*$ group
- $-1 \in G$

Example: $\mathbb{U}_0 = (\mathbb{Z}, \{-1, 1\})$
\textbf{\(\mathbb{P}\)-matroid}

\textbf{Definition:}
(Weak) \(\mathbb{P}\)-matrix:

- \(r \times E\) matrix \(A\) over \(R\)
- \(\det(D) \in G \cup 0\), \(\forall\) \(r \times r\) submatrix \(D\), not all 0
**P-matroid**

**Definition:**
(Weak) P-matrix:

- $r \times E$ matrix $A$ over $R$
- $\det(D) \in G \cup 0$, $\forall$ $r \times r$ submatrix $D$, not all 0

**Theorem (Pendavingh, vZ 2008):**

$$\{ B \subseteq E : |B| = r, \det(A[r, B]) \neq 0 \}$$

is set of bases of matroid $M$.

(Strengthening of [Semple, Whittle 1996])
Homomorphisms

Definition:
Ring homomorphism $\varphi : R_1 \rightarrow R_2$

- $\varphi(1) = 1$
- $\varphi(p)\varphi(q) = \varphi(pq)$
- $\varphi(p) + \varphi(q) = \varphi(p + q)$
Homomorphisms

Definition:
Ring homomorphism $\varphi : R_1 \rightarrow R_2$

- $\varphi(1) = 1$
- $\varphi(p)\varphi(q) = \varphi(pq)$
- $\varphi(p) + \varphi(q) = \varphi(p + q)$

Lemma.

$$\det(\varphi(A)) = \varphi(\det(A))$$
$$\det(\varphi(A)) = 0 \iff \det(A) = 0$$
Theorem:

\[ \{ B \subseteq E : |B| = r, \det(A[r, B]) \neq 0 \} \]

is set of bases of matroid \( M \).
Axiom of choice warning
Proof

Lemma:

$R$ commutative ring

- $R$ has maximal ideal $I$
- $R/I$ is a field
Proof

Lemma:
$R$ commutative ring

- $R$ has maximal ideal $I$
- $R/I$ is a field

Homomorphism $\varphi : R \to R/I$
Proof

Lemma:
$R$ commutative ring
- $R$ has maximal ideal $I$
- $R/I$ is a field

Homomorphism $\varphi : R \rightarrow R/I$

$$\{B \subseteq E : |B| = r, \det(A[r,B]) \neq 0\}$$
Proof

Lemma:
$R$ commutative ring
- $R$ has maximal ideal $I$
- $R/I$ is a field

Homomorphism $\phi : R \to R/I$

$$\{B \subseteq E : |B| = r, \det(A[r,B]) \neq 0\}$$

$$= \{B \subseteq E : |B| = r, \phi(\det(A[r,B])) \neq 0\}$$
Proof

Lemma: 
$R$ commutative ring
- $R$ has maximal ideal $I$
- $R/I$ is a field

Homomorphism $\varphi : R \to R/I$

$$\{B \subseteq E : |B| = r, \det(A[r, B]) \neq 0\}$$

$$= \{B \subseteq E : |B| = r, \varphi(\det(A[r, B])) \neq 0\}$$

$$= \{B \subseteq E : |B| = r, \det(\varphi(A[r, B])) \neq 0\}$$
Many representations from one

Theorem. If

- $M$ is representable over $(R_1, G_1)$
- $\varphi : R_1 \rightarrow R_2$
- $\varphi(G_1) \subseteq G_2$

then $M$ is representable over $(R_2, G_2)$
Many representations from one

**Theorem.** If

- $M$ is representable over $(R_1, G_1)$
- $\varphi : R_1 \to R_2$
- $\varphi(G_1) \subseteq G_2$

then $M$ is representable over $(R_2, G_2)$

\[
\{ B \subseteq E : |B| = r, \det(A[r, B]) \neq 0 \}
= \{ B \subseteq E : |B| = r, \varphi(\det(A[r, B])) \neq 0 \}
= \{ B \subseteq E : |B| = r, \det(\varphi(A[r, B])) \neq 0 \}
\]
Application: universal partial fields

Theorem (Pendavingh, vZ 2009): For every $M$ there exist $P_M$ and $A_M$ such that

$$\exists \varphi : P_M \rightarrow P \text{ with } \varphi(A_M) \approx A$$

for every $P$-representation $A$ of $M$. 
Example

$\mathcal{P}_{\text{Betsy Ross}} = \mathcal{G}$, golden ratio partial field
Example

$P_{P_8} = \mathbb{D}$, dyadic partial field
Dyadic matroids

Theorem (Whittle 1997):
Equivalent for matroid $M$:

(i) $M$ representable over $\text{GF}(3)$ and $\text{GF}(5)$
(ii) $M$ representable over $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \{\pm 2^k\})$
(iii) $M$ representable over $\mathbb{F}$ unless $\chi(\mathbb{F}) = 2$
One representation from many

\[ P_1 \times P_2 := (R_1 \times R_2, G_1 \times G_2) \]

**Lemma (Pendavingh, vZ 2008):**
Matroid representable over both \( P_1 \) and \( P_2 \)
\[ \iff \]
Matroid representable over \( P_1 \times P_2 \)
Application: Lift Theorem

Theorem (Pendavingh, vZ 2008):
Partial fields $\mathcal{P}$, $\hat{\mathcal{P}}$
Homomorphism $\phi : \hat{\mathcal{P}} \to \mathcal{P}$
If bijection between representations over $\mathcal{P}$ and $\hat{\mathcal{P}}$ of

\[
\left\{ \begin{array}{c}
\{ M : E(M) \to \mathcal{P} \} \\
\{ M : E(M) \to \hat{\mathcal{P}} \}
\end{array} \right. 
\]

then

\[
\left\{ \begin{array}{c}
\{ M : E(M) \to \mathcal{P} \} \\
\{ M : E(M) \to \hat{\mathcal{P}} \}
\end{array} \right. = \left\{ \begin{array}{c}
\{ M : E(M) \to \mathcal{P} \} \\
\{ M : E(M) \to \hat{\mathcal{P}} \}
\end{array} \right.
\]
Application: Lift Theorem

Theorem (Pendavingh, vZ 2008):
Partial fields $\mathbb{P}, \mathbb{P}$
Homomorphism $\varphi : \mathbb{P} \rightarrow \mathbb{P}$
If bijection between representations over $\mathbb{P}$ and $\mathbb{P}$ of

then

\[
\left\{ M : E(M) \rightarrow \begin{array}{c} \mathbb{P} \\ \end{array} \right\} = \left\{ M : E(M) \rightarrow \begin{array}{c} \mathbb{P} \\ \end{array} \right\}
\]

Proof generalizes Gerards’ (1989) proof of excluded minors for regular matroids
Examples

(Whittle 1997) Equivalent are:

- Representable over GF(3) and GF(5)
- Representable over $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \{\pm 2^k\})$
Examples

(Whittle 1997) Equivalent are:

- Representable over GF(3) and GF(5)
- Representable over $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \{\pm 2^k\})$

(Whittle 1997) Equivalent are:

- Representable over GF(3) and GF(8)
- Representable over $\mathbb{U}_1 = (\mathbb{Q}(\alpha), \{\pm \alpha^k(1 - \alpha)^l\})$
Examples

(Whittle 1997) Equivalent are:
- Representable over $\text{GF}(3)$ and $\text{GF}(5)$
- Representable over $\mathbb{D} = (\mathbb{Z}[\frac{1}{2}], \{\pm 2^k\})$

(Whittle 1997) Equivalent are:
- Representable over $\text{GF}(3)$ and $\text{GF}(8)$
- Representable over $\bigcup_1 = (\mathbb{Q}(\alpha), \{\pm \alpha^k(1 - \alpha)^l\})$

(Pendavingh, vZ 2008) Equivalent are:
- Representable over $\text{GF}(4)$ and $\text{GF}(5)$
- Representable over $\mathbb{G} = (\mathbb{R}, \{\pm \tau^k\})$

where $\tau$ is golden ratio, root of $x^2 - x - 1$
Two conjectures

Conjecture
Equivalent are:

- Representable over $\text{GF}(2^k)$ for $k \geq 2$
- Representable over $U_{1}^{(2)} = (\text{GF}(2)(\alpha), \{\alpha^k(1 + \alpha)^l\})$
Two conjectures

Conjecture
Equivalent are:

• Representable over $\text{GF}(2^k)$ for $k \geq 2$
• Representable over

$$\mathbb{U}_1^{(2)} = (\text{GF}(2)(\alpha), \{\alpha^k(1 + \alpha)^l\})$$

Conjecture
Equivalent are:

• Representable over $\mathbb{F}$ if $|\mathbb{F}| \geq 4$
• Representable over

$$\mathbb{P}_4 = (\mathbb{Q}(\alpha), \{\alpha^k(1 - \alpha)^l(1 + \alpha)^m(2 - \alpha)^n\})$$
Application: excluded minors for near-regular

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[ M : E(M) \rightarrow \{ \text{GF}(3), \text{GF}(4), \text{GF}(5), \text{GF}(7), \ldots \} \]
namely

\[\begin{cases}
\ldots, & \ldots, & \left(\begin{array}{c}
\bullet
\end{array}\right)^*,
\end{cases}\]

\[\begin{cases}
\ldots, & \left(\begin{array}{c}
\bullet
\end{array}\right)^*,
\end{cases}\]

\[\begin{cases}
\ldots, & \left(\begin{array}{c}
\bullet
\end{array}\right)^*,
\end{cases}\]

\[\begin{cases}
\ldots, & \left(\begin{array}{c}
\bullet
\end{array}\right)^*,
\end{cases}\]
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[
\left\{ M : E(M) \rightarrow \begin{array}{c}
GF(3) \\
GF(4) \\
GF(5) \\
GF(7) \\
\vdots
\end{array} \right\}
\]
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[ M : E(M) \rightarrow \bigcup_1 \]
Part II
Skew Partial Fields
Abandoning commutativity

Until now:

- Rings commutative
- All matroids representable over some field
Abandoning commutativity

Until now:

- Rings commutative
- All matroids representable over some field

Definition:

Skew partial field \( \mathbb{P} := (R, G) \)

- \( R \) ring
- \( G \subseteq R^* \) group
- \(-1 \in G\)
Abandoning commutativity

Until now:

- Rings commutative
- All matroids representable over some field

**Definition:**

*Skew partial field* \( \mathcal{P} := (R, G) \)

- \( R \) ring
- \( G \subseteq R^* \) group
- \(-1 \in G\)

Big problem: **Determinants**
What Would Tutte Do?
Chain groups

**Definition:** $R$ ring, $E$ finite set. *Chain group* is

$$C \subseteq R^E$$

such that, for $c, d \in C$ and $r \in R$

- $0 \in C$
- $c + d \in C$
- $rc \in C$
Chain groups

**Definition:** $R$ ring, $E$ finite set. Chain group is

$$C \subseteq R^E$$

such that, for $c, d \in C$ and $r \in R$

- $0 \in C$
- $c + d \in C$
- $rc \in C$

**Definition:** Support of a chain $c$:

$$\|c\| := \{e \in E : c_e \neq 0\}$$
Chain groups

**Definition:** \( R \) ring, \( E \) finite set. Chain group is
\[
C \subseteq R^E
\]
such that, for \( c, d \in C \) and \( r \in R \)
- \( 0 \in C \)
- \( c + d \in C \)
- \( rc \in C \)

**Definition:** Support of a chain \( c \):
\[
\|c\| := \{ e \in E : c_e \neq 0 \}
\]

**Definition:** Elementary chain: \( c \neq 0 \), inclusionwise minimal support.
Chain groups

**Definition:** *Skew partial field* $\mathbb{P} := (R, G)$

- $R$ ring
- $G \subseteq R^*$ group
- $-1 \in G$
Chain groups

Definition: Skew partial field \( \mathbb{P} := (R, G) \)

- \( R \) ring
- \( G \subseteq R^* \) group
- \(-1 \in G\)

Definition: \( G \)-primitive chain: \( c \in (G \cup \{0\})^E \).
**Chain groups**

**Definition:** *Skew partial field* $\mathbb{P} := (R, G)$

- $R$ ring
- $G \subseteq R^*$ group
- $-1 \in G$

**Definition:** *$G$-primitive* chain: $c \in (G \cup \{0\})^E$.

**Definition:** Chain group is $\mathbb{P}$-chain group if, for all $c \in C$ elementary,

$$c = rd$$

where $r \in R$ and $d \in C$ is $G$-primitive.
Chain groups

Theorem (Pendavingh, vZ 2009):
For a $\mathcal{P}$-chain group $C$, define

$$\mathcal{C}^* := \{||c|| : c \in C, \text{elementary}\}$$

Then $\mathcal{C}^*$ is set of cocircuits of a matroid, $M(C)$. 

**Chain groups**

**Theorem (Pendavingh, vZ 2009):**
For a $\mathcal{P}$-chain group $C$, define

$$C^* := \{ \|c\| : c \in C, \text{elementary} \}$$

Then $C^*$ is set of cocircuits of a matroid, $M(C)$.

**Proof:** Let $e \in \|c\| \cap \|c'\|$. Assume $c, c'$ $G$-primitive. Then $(c_e)^{-1}c - (c'_e)^{-1}c' \in C. \square$
Representability redefined

**Definition:**
A matroid $M$ is $\mathbb{P}$-representable if there is a $\mathbb{P}$-chain group with $M = M(C)$. 
Duality

Definition:

*Opposite* of a ring $R = (S, 0, 1, +, \cdot)$ is

$$R^\circ := (S, 0, 1, +, \circ)$$

with $r \circ s := s \cdot r$. 
Duality

**Definition:**
*Opposite* of a ring $R = (S, 0, 1, +, \cdot)$ is

$$R^\circ := (S, 0, 1, +, \circ)$$

with $r \circ s := s \cdot r$.

**Definition:**
*Opposite* partial field is

$$\mathbb{P}^\circ := (R^\circ, G^\circ).$$
Duality

Definition: Opposite of a ring $R = (S, 0, 1, +, \cdot)$ is

$$R^\circ := (S, 0, 1, +, \circ)$$

with $r \circ s := s \cdot r$.

Definition: Opposite partial field is

$$\mathbb{P}^\circ := (R^\circ, G^\circ).$$

Theorem: $M$ is $\mathbb{P}$-representable if and only if $M^\ast$ is $\mathbb{P}^\circ$-representable
Generator matrix

- Pick basis $B$
- $B$-fundamental cocircuits: $C_{B,e}$
- Pick $G$-primitive chains $a^e$ for $C_{B,e}$
- Let $A$ be matrix with rows $a^e$

**Theorem (Pendavingh, vZ 2009):**
The row span of $A$ equals the chain group.

**Theorem (Pendavingh, vZ 2009):**
$B$ is a basis if and only if $A[r, B]$ is invertible.
Tutte’s Representability Theorem

Definition:
Cocircuits $D_1, \ldots, D_k$ are modular if

$$\text{rk}(M.(D_1 \cup \cdots \cup D_k)) = 2$$

Theorem (Pendavingh, vZ, after Tutte):
For each $D \in C^*$, pick $G$-primitive chain $\alpha^D$. Let $C$ be chain group thus generated.
Then $M = M(C)$ if and only if $\exists p, p', p'' \in G$

$$p\alpha^D + p'\alpha^{D'} + p''\alpha^{D''} = 0$$

whenever $D, D', D''$ are modular.
Part III

A matroid
Example

Quaternion group $H$:

- 8 elements: $\{1, i, j, k, -1, -i, -j, -k\}$
- Relations $i^2 = j^2 = k^2 = ijk = -1$

Quaternions $\mathbb{H} := \mathbb{R}[i, j, k]$ form a skew field.
Example

Quaternion group \( H \):

- 8 elements: \( \{1, i, j, k, -1, -i, -j, -k\} \)
- Relations \( i^2 = j^2 = k^2 = ijk = -1 \)

Quaternions \( \mathbb{H} := \mathbb{R}[i, j, k] \) form a skew field.

\( Q_3(H) \) is rank-3 Dowling geometry of \( H \).

Representation: \( \mathbb{H} \)-chain group is generated by

\[
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & i & \cdots & -1 & 0 & -k \\
0 & 1 & 0 & 1 & -1 & 0 & i & -1 & 0 & & -k & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 & 0 & i & -1 & & 0 & -k & -1
\end{bmatrix}
\]
Example

Theorem (Pendavingh, vZ 2009):
The matroid $Q_3(H) \oplus PG(3, 3)$ is representable over a skew partial field $\mathbb{P}$ if and only if there is a (ring) homomorphism

$$(GF(3)[i, j, k], H) \rightarrow \mathbb{P}$$
Example

**Theorem (Pendavingh, vZ 2009):**
The matroid $Q_3(H) \oplus PG(3, 3)$ is representable over a skew partial field $\mathbb{P}$ if and only if there is a (ring) homomorphism

$$(GF(3)[i, j, k], H) \rightarrow \mathbb{P}$$

**Corollary (Pendavingh, vZ 2009):**
This matroid is not representable over any skew field!
Example

Theorem (Pendavingh, vZ 2009):
The matroid \( Q_3(H) \oplus PG(3, 3) \) is representable over a skew partial field \( \mathbb{P} \) if and only if there is a (ring) homomorphism

\[
(GF(3)[i, j, k], H) \rightarrow \mathbb{P}
\]

Corollary (Pendavingh, vZ 2009):
This matroid is not representable over any skew field!

Remark:
Usual suspects (Vámos, non-Desargues) still not representable.
Open problems

**Problem:** Find small, 3-connected $\mathbb{P}$-representable matroids not representable over any field.

**Problem:** For which groups $G$ are the Dowling geometries $\mathbb{P}$-representable for some skew partial field?

**Problem:** Does Ingleton’s Inequality hold for skew-partial-field-representable matroids?
Slides, preprints at http://www.cwi.nl/~zwam/
Copies of thesis available!
The End