Stability, Fragility, and Rota’s Conjecture

Stefan van Zwam

Based on joint work with Carolyn Chun, Rhiannon Hall, Dillon Mayhew, Rudi Pendavingh, and Geoff Whittle

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Theorem (Mayhew, Whittle, vZ 2010). Rota’s Conjecture for GF(5) is implied by the Bounded Canopy Conjecture.
Ingredients

I. Rota’s Conjecture
II. Stabilizers
III. Fragility
IV. Blocking Sequences
Part I

Rota’s Conjecture
Matroid representation

Definition. Representation of matroid $M$ over (partial) field $\mathbb{F}$: dependency-preserving map $E(M) \to \mathbb{F}^r$

Problem. How to show no such map exists?
Minors

- **Deletion**: $M \setminus e := (E - \{e\}, \{I \in \mathcal{I} : e \notin I\})$
- **Contraction**: $M/e := (E - \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\})$
- **Minors**: Obtained from sequence of such steps
Minors

- **Deletion**: \( M \setminus e := (E - \{ e \}, \{ I \in \mathcal{I} : e \notin I \}) \)
- **Contraction**: \( M/e := (E - \{ e \}, \{ I : I \cup \{ e \} \in \mathcal{I} \}) \)
- **Minors**: Obtained from sequence of such steps
  - Generate partial order
  - Preserve representability
Excluded minors
Rota’s Conjecture

Conjecture (Rota 1971): If $\mathbb{F}$ finite, then $\exists k = k(\mathbb{F})$: exactly $k$ excluded minors for

\[
\left\{ M : E(M) \rightarrow \mathbb{F} \right\}
\]
Rota’s Conjecture

Conjecture (Rota 1971): \( F \) finite, then \( \exists k = k(F) \): exactly \( k \) excluded minors for

\[
\left\{ M : E(M) \rightarrow F \right\}
\]

Proven for:

- GF(2): Tutte (1958)
- GF(3): Bixby (1979), Seymour (1979)
Rota’s Conjecture

Conjecture (Rota 1971): If $\mathbb{F}$ finite, then $\exists k = k(\mathbb{F})$: exactly $k$ excluded minors for

$$\left\{ M : E(M) \rightarrow \mathbb{F} \right\}$$

- $\mathbb{GF}(2)$: 1
- $\mathbb{GF}(3)$: 4
- $\mathbb{GF}(4)$: 7
- $\mathbb{GF}(5)$: $\geq 564$ (Mayhew, Royle ’09)
Sketch of a proof

(i) Take an excluded minor $M$

(ii) Show it’s not too big

Definition.

$u, v \subseteq E(M)$ is deletion pair if

• $\text{rk}^*(\{u, v\}) = 2$

• ...
Sketch of proof

\[ E(M \setminus \nu) \rightarrow A_1 \]

\[ E(M \setminus u) \rightarrow A_2 \]
Sketch of proof

\[ E(M \setminus \nu) \rightarrow A' \]

\[ E(M \setminus u) \rightarrow A' \]

\[ E(M \setminus \nu) \rightarrow E(M \setminus \nu) \rightarrow \]

\[ A' \]

\[ \chi \]

\[ \chi' \]

\[ u \quad \nu \]
Sketch of proof

\[ A = A' \]

[Diagram showing a rectangular matrix with parts labeled u, x, v, and x']
Sketch of proof

\[
A = A' \\
M \neq M[A]
\]
Sketch of proof

\[ A = B \text{ basis in one of } M, M[A] \]
Sketch of proof

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Sketch of proof

$A = \begin{pmatrix} 1 & \ast & \ast \\ a & 1 & \ast \\ b & \ast & \ast \end{pmatrix}$

$B$ basis in one of $M, M[A]$
Sketch of proof
Sketch of proof

GF(2):

\[
\begin{array}{cc}
  a & u \\
  b & v \\
\end{array}
\]

GF(3):

\[
\begin{array}{ccc}
  a & u & v \\
  b & * & * \\
\end{array}
\]

\[
\begin{array}{cc}
  a & * \\
  b & * \\
\end{array}
\]

\[
\begin{array}{cc}
  a & * \\
  b & 0 \\
\end{array}
\]

\[
\begin{array}{cc}
  a & * \\
  b & 0 \\
\end{array}
\]

\[
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\end{array}
\]
Part II
Stabilizers
Sketch of proof

\[ E(M \setminus v) \rightarrow A_1 \]

\[ E(M \setminus u) \rightarrow A_2 \]
Sketch of proof

\[ E(M \setminus \nu) \rightarrow A' \]

\[ E(M \setminus u) \rightarrow A' \]
Stabilizers

Definition.

$N$ stabilizes $M$ over $\mathbb{F}$ if $N \leq M$ and representations of $N$ extend to at most one representation of $M$. 
Stabilizers

Definition.
\( N \) stabilizes \( M \) over \( \mathbb{F} \) if \( N \preceq M \) and representations of \( N \) extend to at most one representation of \( M \).

Definition.
\( N \) strongly stabilizes \( M \) over \( \mathbb{F} \) if \( N \preceq M \) and representations of \( N \) extend to exactly one representation of \( M \).
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**Definition.**

\( N \) (strongly) stabilizes \( \mathcal{M} \) if \( N \) (strongly) stabilizes all 3-connected members in \( \mathcal{M} \) with \( N \)-minor.
Stabilizers

Theorem (Kahn 1988).
$U_{2,4}$ is a strong stabilizer for $\mathcal{M}(GF(4))$.

Theorem (Whittle 1999).
$U_{2,5}, U_{3,5}$ are stabilizers for $\mathcal{M}(GF(5))$. 
Deletion pairs, updated

**Definition.**

\( u, v \subseteq E(M) \) is deletion pair preserving \( N \) if

- \( \text{rk}^*(\{u, v\}) = 2 \)
- \( M \setminus u, v \) has \( N \)-minor
- \( \text{co}(M \setminus u), \text{co}(M \setminus v), \text{co}(M \setminus u, v) \) are 3-connected
Part III
Fragility
**Fragility**

**Definition.**

\( M \) is \( N \)-fragile if \( \forall e \in E(M) \) one of \( M \setminus e, M/e \) has no \( N \)-minor.

*Strictly:* \( M \) has \( N \)-minor.
Fragility

Basic properties of $N$-fragile matroid $M$

- $M$ is 3-connected up to parallel and series classes
- Minor-closed
Fragility

Basic properties of $N$-fragile matroid $M$

- $M$ is 3-connected up to parallel and series classes
- Minor-closed

Example.
$U_{2,4}$-fragile, simple, cosimple matroid in GF(4) is $U_{2,5}$, $U_{3,5}$, or a whirl.
Bounded Canopy Conjecture

Conjecture (Geelen, Gerards, Whittle 2006).

\[ \exists k = k(N, F) : \]

If \( M \) is \( F \)-representable, strictly \( N \)-fragile then

\[ \text{branch width}(M) \leq k \]
**Bounded Canopy Conjecture**

**Conjecture (Geelen, Gerards, Whittle 2006).**

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If \( M \) is \( F \)-representable, strictly \( N \)-fragile then

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**Theorem (Robertson, Seymour XXII)**

It’s true for graphs.
Bounded Canopy Conjecture

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Theorem (Robertson, Seymour XXII)
It’s true for graphs.

Theorem (Geelen, Whittle 2002).
\[ \forall k \in \mathbb{N}: \text{finitely many excluded minors for } M(GF(q)) \]
have branch width \( k \).
Part IV

Blocking Sequences
Connectivity

Need:

- essentially 3-connected minor with $N, a, b, u$
- essentially 3-connected minor with $N, a, b, v$
- together bounded branch width
Blocking Sequences

- $N'$ is strictly $N$-fragile matroid in $M\setminus u, v$
- Add back $a, b, u$ or $v$: 2-separations
- Fix these with *blocking sequence*
Blocking Sequences

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- Add back $a, b, u$ or $v$: 2-separations
- Fix these with blocking sequence

Theorem (Geelen, Hliněný, Whittle 2004).
If $\{v_1, \ldots, v_t\}$ is blocking sequence for $k$-separation $(A, B)$ then adding back $\{v_1, \ldots, v_t\}$ increases branch width by at most $k$. 


Conclusion

Theorem (Mayhew, Whittle, vZ 2010).

\( \mathbb{F} \) finite field, \( N \) matroid such that

(i) \( N \) is 3-connected, not binary

(ii) \( N \) stabilizes \( \mathcal{M}(\mathbb{F}) \)

If Bounded Canopy Conjecture is true for \( \mathbb{F} \), then finitely many excluded minors for \( \mathcal{M} \) have \( N \)-minor.
Explicit computations

**Theorem (Geelen, Gerards, Kapoor 2000).**
Exactly 7 excluded minors for $\mathcal{M}(\text{GF}(4))$.

**Theorem (Hall, Mayhew, vZ 2009).**
Exactly 10 excluded minors for near-regular matroids
namely

\[ \{ \ldots, \quad \Delta Y \quad \} \]
Quinary matroids

Theorem (Pendavingh, vZ 2008).

- 1 quinary representations $\Rightarrow \mathbb{H}_1$-representable
- 2 quinary representations $\Rightarrow \mathbb{H}_2$-representable
- 3 quinary representations $\Rightarrow \mathbb{H}_3$-representable
- 4 quinary representations $\Rightarrow \mathbb{H}_4$-representable
- 5 quinary representations $\Rightarrow \mathbb{H}_6$-representable

where $\mathbb{H}_k$ is partial field: subdeterminants of representation are in $R^* \cup \{0\}$. 
Take-home message

Need to study structure and branch width of $N$-fragile matroids.

(ONGOING WORK WITH CHUN, MAYHEW, WHITTLE: $\{F_7^-, (F_7^-)^*, P_8\}$-fragile, $\{U_{3,5}, U_{2,5}\}$-fragile inside appropriate classes)
Slides, preprints at http://www.cwi.nl/~zwam/
Copies of thesis available!
The End