Beyond Total Unimodularity

Stefan van Zwam

Based on joint work with Rhiannon Hall, Dillon Mayhew, Rudi Pendavingh, and Geoff Whittle

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An optimization problem
An optimization problem

minimize $2x_{11} + 3x_{12} + 4x_{13} + 2x_{21} + 2x_{22} + 8x_{23}$

such that

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_{11} \\
x_{12} \\
x_{13} \\
x_{21} \\
x_{22} \\
x_{23}
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
6 \\
2 \\
5 \\
3
\end{bmatrix}
$$
An optimization problem
An optimization problem

**Theorem.**
If matrix *totally unimodular*, then integer optimal solution.
Totally unimodular matrices

Definition.
A matrix is *totally unimodular* if for every square submatrix $D$

$$\det(D) \in \{-1, 0, 1\}$$
Testing for total unimodularity

Theorem (Truemper 1982).
There is a polynomial-time algorithm to test if a matrix $A$ over $\mathbb{R}$ is totally unimodular.
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Main ingredient:
Seymour’s Decomposition Theorem for Regular Matroids(1980).
Overview

Total unimodularity
and its (im)possible generalizations:

I. Matroids and representations
II. Seymour’s Decomposition Theorem
III. Excluded minors
Part I
Matroids and representations
Matroids
Matroids
Matroids
Matroid axioms

Definition.
Given

- $E$: finite set
- $\mathcal{I}$: collection of subsets

such that

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

  $$\exists e \in J - I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

Then $M = (E, \mathcal{I})$ is a matroid.
Example: the Fano matroid

- $E = \{ \text{points} \}$
- $\mathcal{I} = \{ X \subseteq E \text{ in general position} \}$
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Representations

Definition.
A representation of $M$ over field $\mathbb{F}$ is a dependency-preserving map

$$A : E(M) \rightarrow \mathbb{F}^r.$$
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Example: the Fano matroid

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} - 
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} - 
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} = -
\begin{bmatrix}
0 \\
0 \\
2
\end{bmatrix}
\]
Representations

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View $A$ as $r \times E$ matrix!
Regular matroids

Theorem (Tutte 1958).
Equivalent for a matroid $M$:

- $M$ representable over all fields
Regular matroids

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Such matroids are called regular.
and beyond

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A matrix is dyadic if every subdeterminant is in

$$\{\pm 2^k : k \in \mathbb{Z}\} \cup \{0\}.$$
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- $M$ representable over $\text{GF}(3)$, $\text{GF}(8)$
- $M$ has *near-regular* representation over $\mathbb{Q}(\alpha)$

A matrix is *near-regular* if every subdeterminant is in

$$\{\pm \alpha^k(1 - \alpha)^l : k, l \in \mathbb{Z}\} \cup \{0\}.$$
...and beyond

Theorem (Vertigan, unpublished; Pendavingh, vZ 2010).
Equivalent for a matroid $M$:

- $M$ representable over $\text{GF}(4)$, $\text{GF}(5)$
- $M$ has *golden ratio* representation over $\mathbb{R}$

A matrix is *golden ratio* if every subdeterminant is in

$$\{\pm \tau^k : k \in \mathbb{Z}\} \cup \{0\}$$

where $\tau$ is the *golden ratio*, i.e. $\tau^2 - \tau - 1 = 0$. 
Part II

Seymour’s Decomposition Theorem
Operations

Elementary operations that preserve T.U.:

- Scale rows and columns by $-1$
- Permute rows and columns
- Row-reduce a column to an identity vector

\[
\begin{bmatrix}
\alpha & c \\
\hline
b & D
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & \alpha^{-1}c \\
\hline
0 & D - b\alpha^{-1}c
\end{bmatrix}
\]
Operations
Dualizing:

\[[I \ A] \rightarrow [\ -A^T \ I']\]
Operations

Operations that preserve T.U.: 1-sums

\[ A_1 \oplus_1 A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \]
Operations

Operations that preserve T.U.: 2-sums

\[
\begin{bmatrix}
A_1 \\
\vdots \\
0 \\
a_1
\end{bmatrix}
\oplus_2
\begin{bmatrix}
1 & a_2 \\
0 & A_2 \\
0 & 0
\end{bmatrix}
=
\begin{bmatrix}
A_1 & 0 \\
a_1 & a_2 \\
0 & A_2
\end{bmatrix}
\]
Operations

Operations that preserve T.U.: 3-sums

\[
\begin{bmatrix}
A_1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
\alpha_1 & 1 & 1 & 0 \\
\beta_1 & 1 & 0 & 1
\end{bmatrix}
\oplus_3
\begin{bmatrix}
1 & 1 & 0 & a_2 \\
1 & 0 & 1 & b_2 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
A_1 & 0 \\
\alpha_1 & a_2 \\
\beta_1 & b_2 \\
0 & A_2
\end{bmatrix}
\]
3-sums geometrically
Have cement, need bricks

Definition.
A matroid is *graphic* if its independent sets are the edge sets of forests of a graph $G$. 
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Representation: incidence matrix. Entries in \{0, 1, -1\} with

- At most two nonzeroes in each column
- If two, then one 1 and one -1
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A matroid is *graphic* if its independent sets are the edge sets of forests of a graph $G$.

Representation: incidence matrix. Entries in $\{0, 1, -1\}$ with

- At most two non-zeroes in each column
- If two, then one $1$ and one $-1$

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
-1 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 \\
\end{bmatrix}
\]
Have cement, need bricks

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
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\end{bmatrix}
\]

**Theorem.**
A graphic matroid is regular.
What else is out there?

Theorem (Seymour 1980).
Suppose $M$, regular, can *not* be obtained from graphic matroids by *dualizing*, 1-sums, 2-sums. Then $M$ has one of the following as minor:

$R_{10}, R_{12}$
The case $R_{10}$

$$\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1 \\
\end{bmatrix}$$

**Theorem.** If $M$ regular, contains $R_{10}$, not equal to $R_{10}$, it can be written as a 1- or 2-sum.
The case $R_{12}$

$$
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & -1 & -1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 \\
\end{bmatrix}
$$

**Theorem.** If $M$ regular and contains $R_{12}$, it can be written as a 3-sum.
Seymour’s Decomposition Theorem

Theorem (Seymour 1980).
Every regular matroid can be obtained from graphic ones and $R_{10}$ by dualizing, $k$-sums for $k = 1, 2, 3$. 
Seymour’s Decomposition Theorem

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A matroid can be tested for being graphic in polynomial time.
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A matroid can be tested for being graphic in polynomial time.

Theorem (Truemper 1982).
A matroid can be tested for being regular in polynomial time.
... and beyond?

**Problem.**
Can a matroid be tested for being *near-regular* in polynomial time?

**Problem.**
Is there a satisfying decomposition theorem for near-regular matroids?
Recognizing signed-graphic matroids

Definition.
A matroid is signed-graphic ⇔ representation over GF(3) with at most 2 nonzero entries per column.
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Theorem (Geelen; Mayhew – unpublished).
There is no polynomial-time algorithm to test if a matroid, given by rank oracle, is signed-graphic.
Recognizing signed-graphic matroids

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But...
Recognizing signed-graphic matroids

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A matroid is signed-graphic \(\iff\) representation over \(\text{GF}(3)\) with at most 2 nonzero entries per column.

Theorem (Geelen; Mayhew – unpublished).
There is no polynomial-time algorithm to test if a matroid, given by rank oracle, is signed-graphic.

But...
What if \(M\) is given as GF(3)-matrix?
Recognizing signed-graphic matroids

‘Theorem’ (Musitelli 2008).
Polynomial-time algorithm to decide if

\[ A = D^{-1}A' \]

with \( A' \) dyadic signed-graphic and \( D \) invertible submatrix of \( A' \)
Recognizing near-regular-graphic matroids

What about decomposition?

Natural condition for decomposition:

- No basic class contains all graphic \textit{and} all co-graphic matroids.

\textbf{Corollary (Mayhew, Whittle, vZ 2011).}
Any natural decomposition of the near-regular matroids must employ 4-sums.
What about decomposition?

**Theorem (Mayhew, Whittle, vZ 2011).**

$M_1, M_2$ graphic matroids. Can build internally 4-connected near-regular matroid having both $M_1$ and dual of $M_2$ in it.

$$A_{12} = \begin{bmatrix}
  d & e & f & 4 & 5 & 6 \\
  a & 1 & 0 & 1 & 1 & 1 & 0 \\
  b & 0 & -1 & 1 & 1 & 0 & \alpha \\
  c & 1 & 1 & 0 & 0 & \alpha & -\alpha \\
  1 & 0 & 0 & 0 & 1 & 0 & 1 \\
  2 & 0 & 0 & 0 & 0 & 1 & -1 \\
  3 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}$$
Part III
Excluded minors
Minors: abstract definition

- **Deletion**: \( M \setminus e := (E - \{e\}, \{I \in \mathcal{I} : e \notin I\}) \)
- **Contraction**: \( M/e := (E - \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\}) \)
- **Minors**: Obtained from sequence of such steps
Minors: abstract definition

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- **Contraction**: \( M/e := (E - \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\}) \)
- **Minors**: Obtained from sequence of such steps
  - Generate partial order
  - Preserve representability
Excluded minors
Regular matroids

Theorem (Tutte 1958):
Exactly 3 excluded minors for regular matroids, namely

\[
\begin{align*}
\{ & , \quad \begin{array}{c} \text{circle} \\ \text{triangle} \end{array}, \quad \left( \begin{array}{c} \text{circle} \\ \text{triangle} \end{array} \right) \ast \end{align*}
\]
Near-regular matroids

**Theorem (Hall, Mayhew, vZ 2011):**
Exactly 10 excluded minors for near-regular matroids,
namely

\[
\{ \cdots, \triangle, (\triangle)^*, m, (\square)^*, (\square)^{\Delta Y}\}.
\]
...and beyond?

Problem.
Is there a finite number of excluded minors for dyadic matroids?
...and beyond?

Problem.
Is there a finite number of excluded minors for dyadic matroids?

Matroid Minors Project (Geelen, Gerards, Whittle, in progress).
Yes! (And an algorithm too!)
Where to learn more?

- Matroïden en hun representaties, Nieuw Archief voor Wiskunde, December 2010


- (With Rhiannon Hall and Dillon Mayhew) The excluded minors for near-regular matroids, European Journal of Combinatorics, in press, 2011

Slides, preprints at http://www.cwi.nl/~zwam/

The End