

Fragility in matroid theory

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Based on joint and ongoing work with Carolyn Chun,
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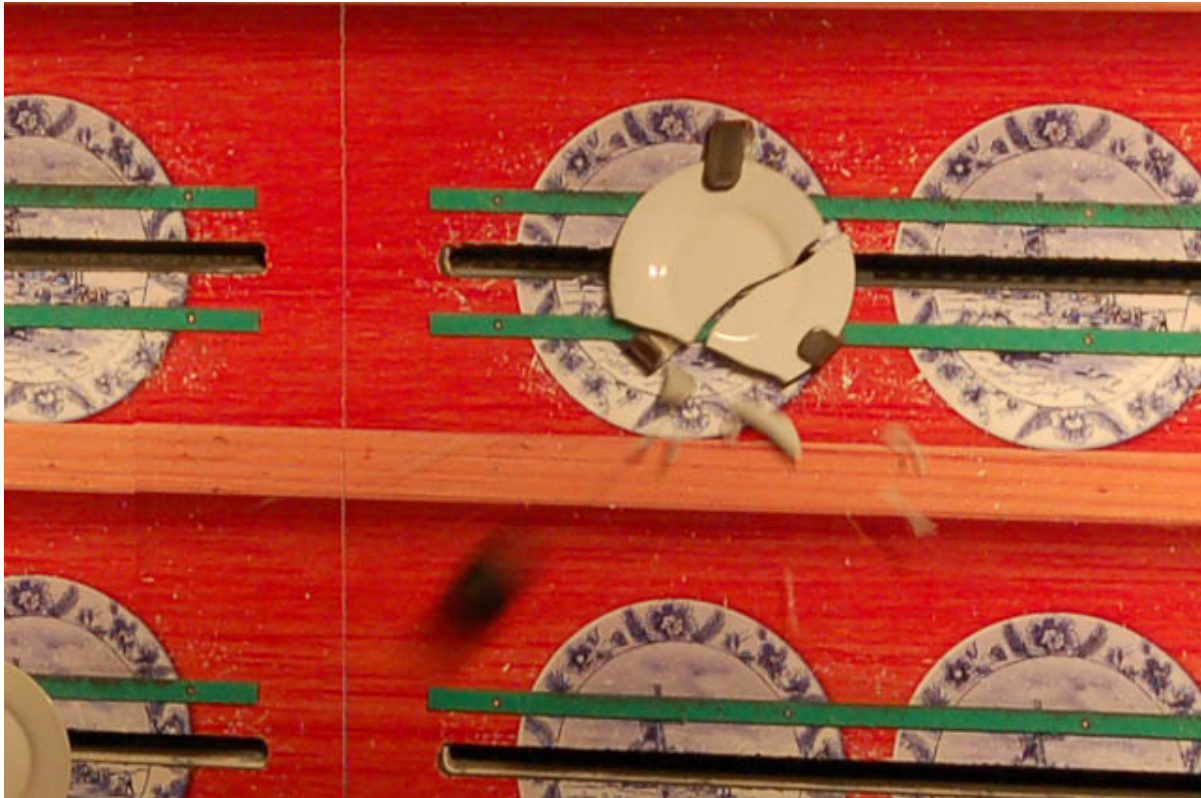
DIAMANT Symposium, Heeze, The Netherlands. May 27, 2011

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Part I

Matroids and fragility



What is a matroid?

ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.¹

By HASSLER WHITNEY.

1. Introduction. Let C_1, C_2, \dots, C_n be the columns of a matrix M . Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

- (a) Any subset of an independent set is independent.
- (b) If N_p and N_{p+1} are independent sets of p and $p + 1$ columns respectively, then N_p together with some column of N_{p+1} forms an independent set of $p + 1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a “matroid.” The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.

What is a matroid?

Definition. Given

E : finite set

\mathcal{I} : collection of subsets

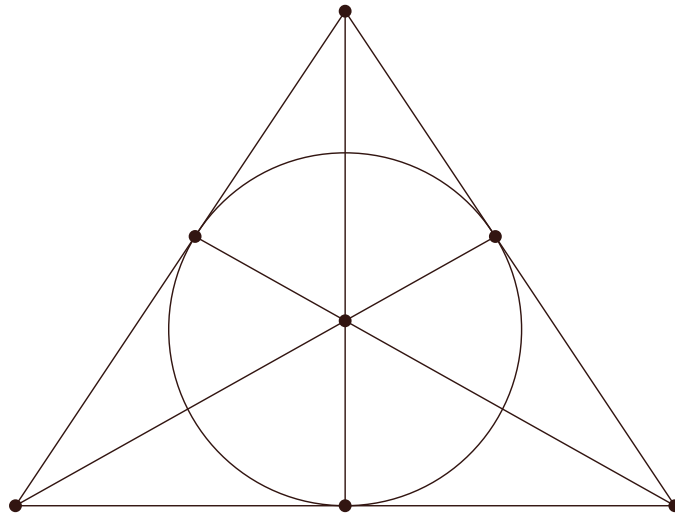
such that

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

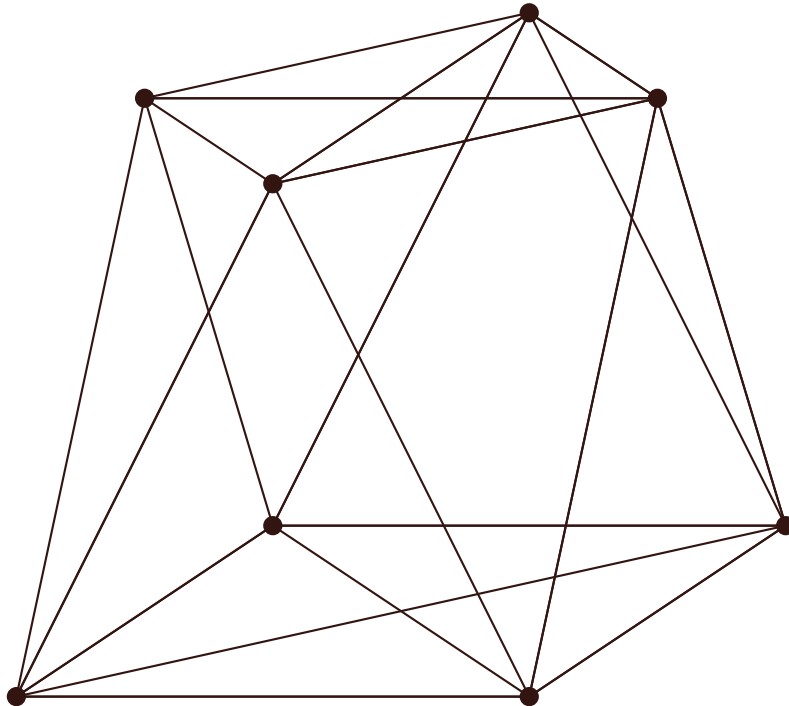
$\exists e \in J - I$ such that $I \cup \{e\} \in \mathcal{I}$

Then $M = (E, \mathcal{I})$ is a **matroid**.

What is a matroid?



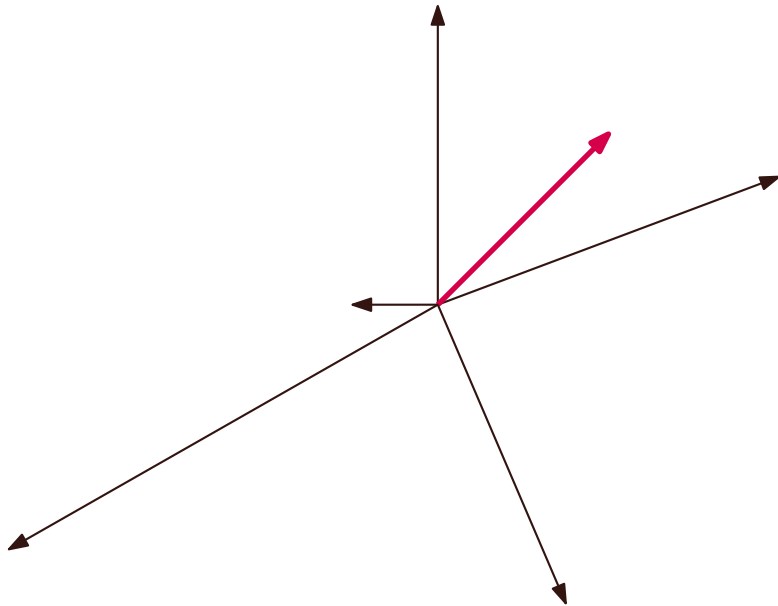
What is a matroid?



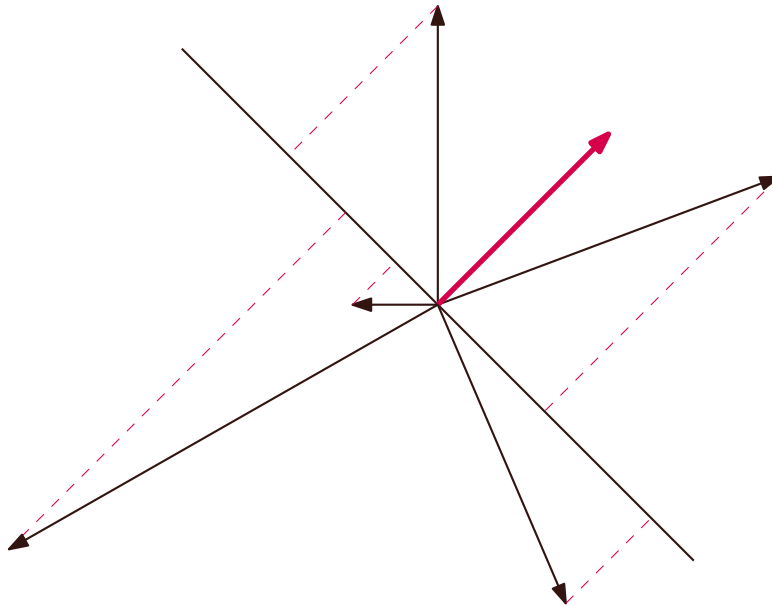
Minors

- *Deletion*: $M \setminus e := (E - \{e\}, \{I \in \mathcal{I} : e \notin I\})$
- *Contraction*: $M / e := (E - \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\})$
- *Minors*: Obtained from sequence of such steps

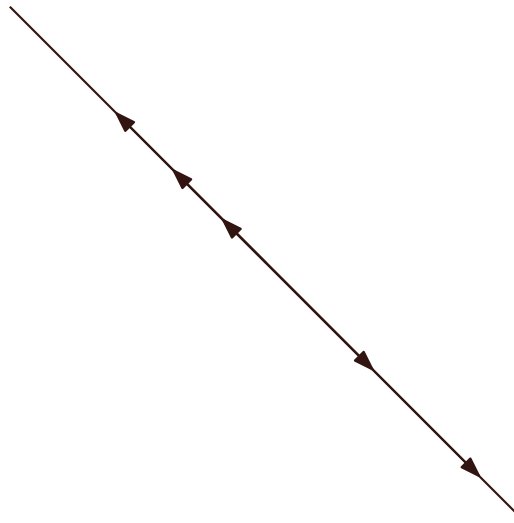
Contraction



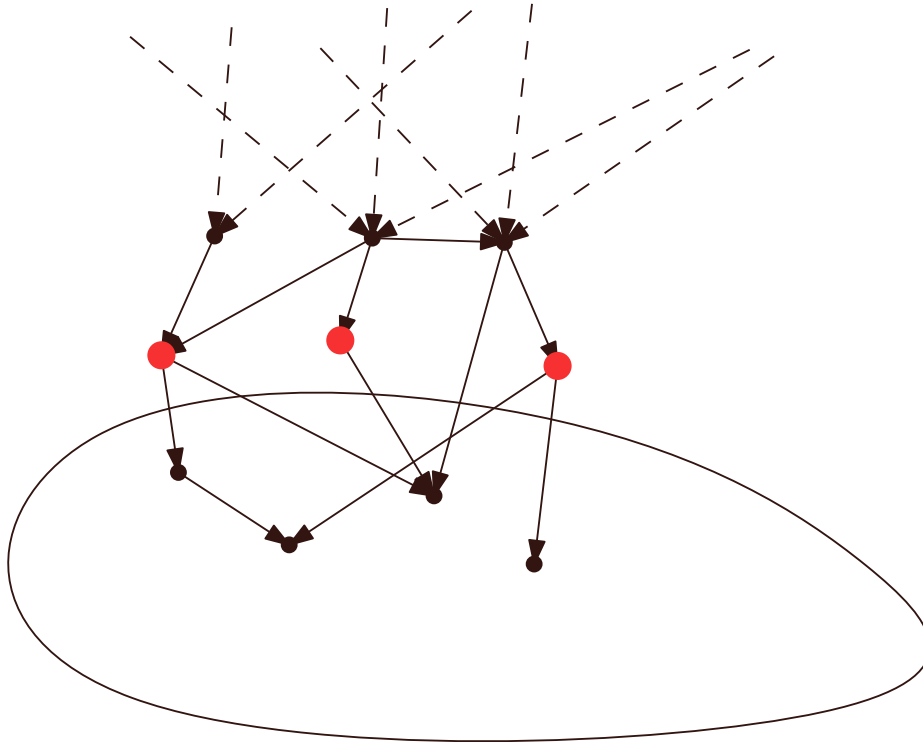
Contraction



Contraction



Minor order



Excluded minors

Definition.

Matroid M is *excluded minor* for minor-closed class \mathcal{C} if

- $M \notin \mathcal{C}$
- For all e : $M \setminus e$ **and** M/e in \mathcal{C}

Fragility

Historical definition.

Matroid M is *almost- \mathcal{C}* for minor-closed class \mathcal{C} if

- $M \notin \mathcal{C}$
- For all e : $M \setminus e$ **or** M/e in \mathcal{C}

Fragility

Definition.

Matroid M is \mathcal{N} -fragile for set of matroids \mathcal{N} if

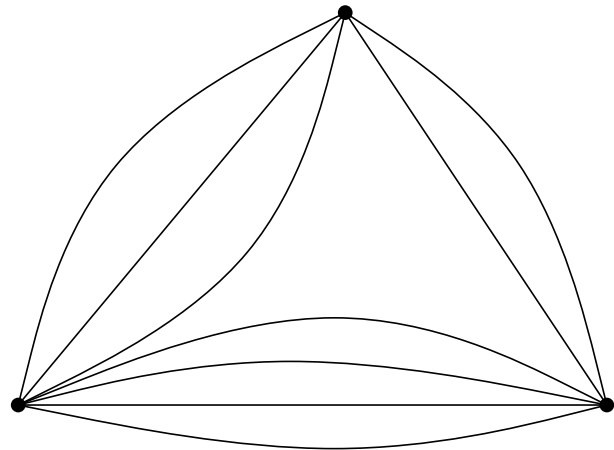
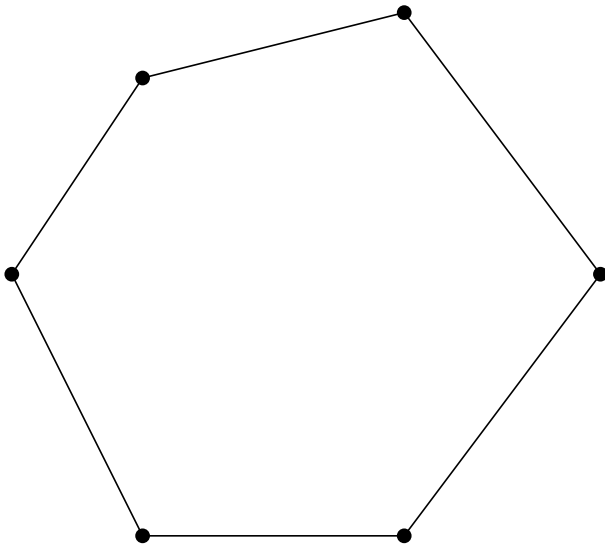
- For all e : $M \setminus e$ **or** M/e has no minor in \mathcal{N}

Example

Let G be a *triangle-fragile* graph.

Example

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Part II

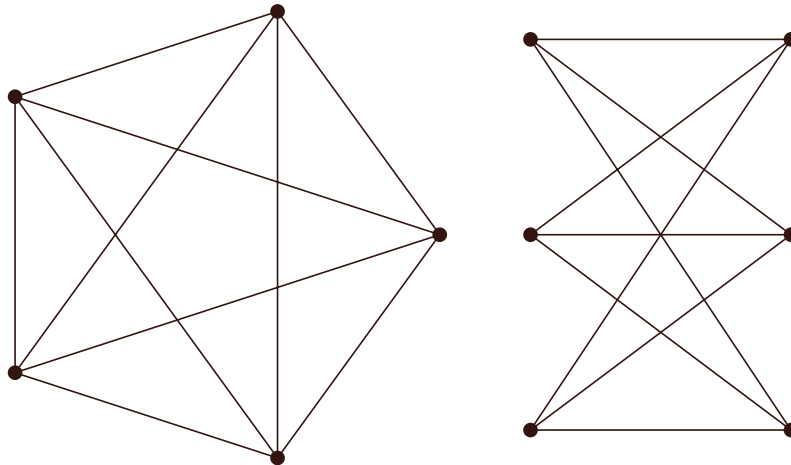
Excluded minors



Kuratowski's Theorem

Theorem.

A graph is planar if and only if it has no minor isomorphic to one of



Wagner's Conjecture

Theorem (Robertson and Seymour, Graph Minors XX)

Let \mathcal{C} be a minor-closed class of graphs. There is a finite number of excluded minors for \mathcal{C} .

Wagner's Conjecture

Theorem (Robertson and Seymour, Graph Minors XX)

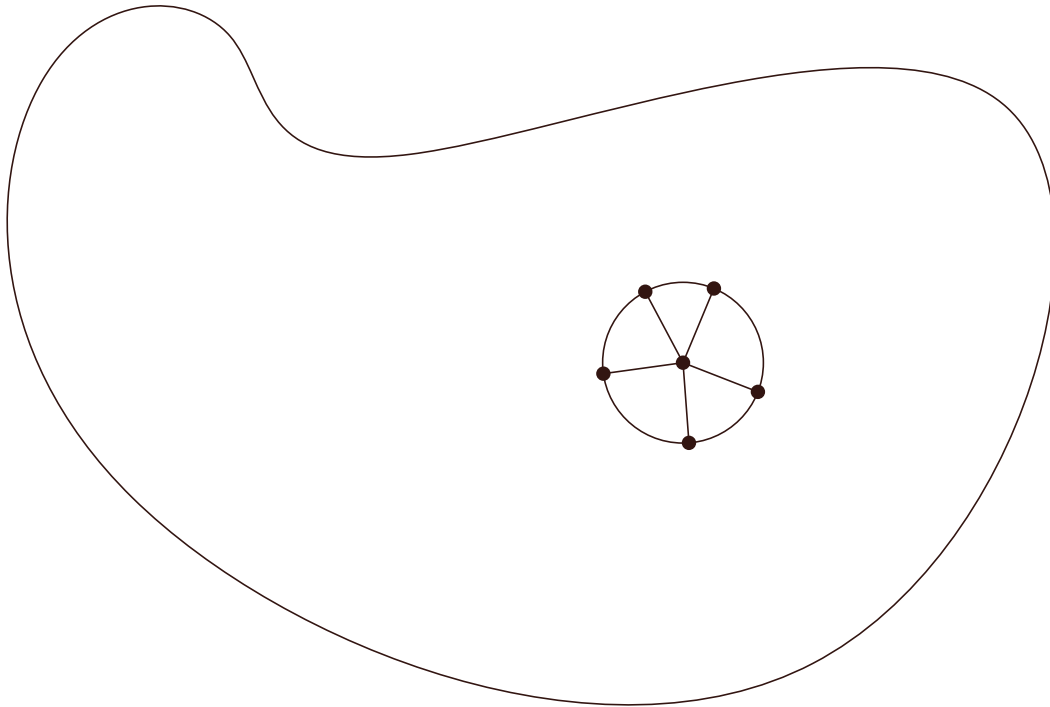
Let \mathcal{C} be a minor-closed class of graphs. There is a finite number of excluded minors for \mathcal{C} .

Theorem (Robertson and Seymour)

There is a polynomial-time algorithm to test if $G \in \mathcal{C}$.

Irrelevant vertex

- Low tree width: dynamic programming
- High tree width: find irrelevant vertex



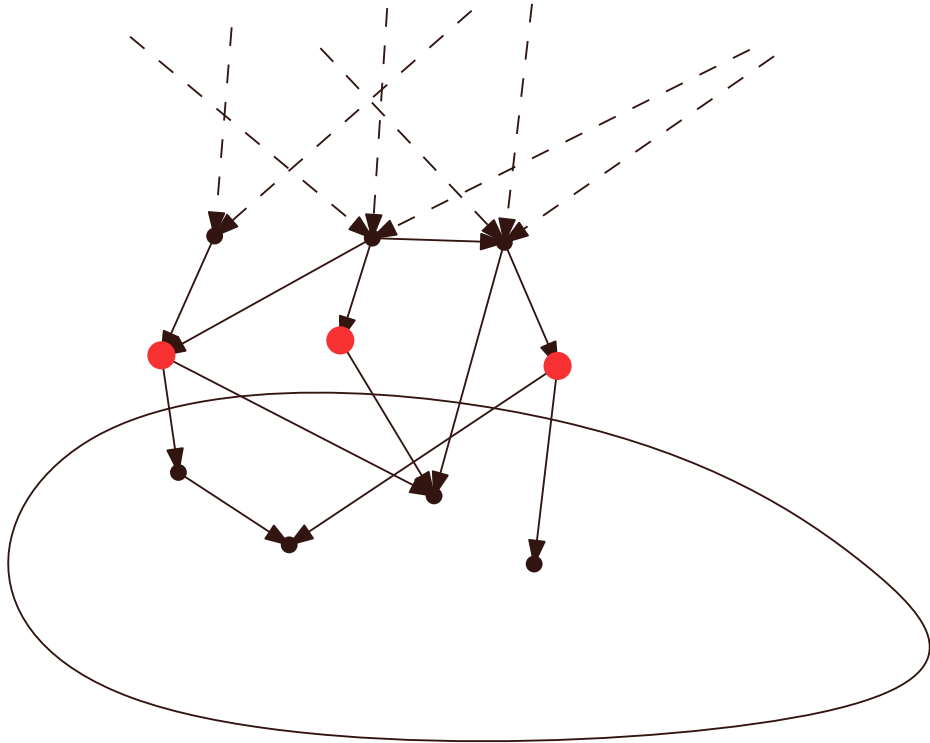
Matroid Minor Project

Bad news. Let \mathcal{C} be all real-representable matroids.

Theorem (Mayhew, Newman, Whittle 2009).

Let M be a real-representable matroid. Then there is an excluded minor for \mathcal{C} having M as minor.

Minor order



Matroid Minor Project

However...

Let \mathcal{C}_q be all matroids representable over $\text{GF}(q)$.

Conjecture (and work in progress by Geelen, Gerards, Whittle).

Let \mathcal{C} be a minor-closed subclass of \mathcal{C}_q . There is a finite number of excluded minors for \mathcal{C} in \mathcal{C}_q .

Matroid Minor Project

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Conjecture (and work in progress by Geelen, Gerards, Whittle).

There is a polynomial-time algorithm to test if $M \in \mathcal{C}$, if representation given.

Flexible element

- Low branch width: dynamic programming.
- High branch width: find *flexible element*: $M \setminus e$ and M/e both have N -minor.

Flexible element

- Low branch width: dynamic programming.
- High branch width: find *flexible element*: $M \setminus e$ and M/e both have N -minor.
- ... in other words: M is not N -fragile

Bounded Canopy Conjecture

Conjecture (Geelen, Gerards, Whittle 2006).

$\exists k = k(N, \mathbb{F}) :$

If M is \mathbb{F} -representable, strictly N -fragile then

$$\text{bw}(M) \leq k$$

Rota's Conjecture

Let \mathcal{C}_q be all matroids representable over $\text{GF}(q)$.

Conjecture (Rota 1971).

There is a finite number of excluded minors for \mathcal{C}_q .

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Let \mathcal{C}_q be all matroids representable over $\text{GF}(q)$.

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There is a finite number of excluded minors for \mathcal{C}_q .

- Matroid Theorists' Holy Grail
- Proven for $q \leq 4$
- Outside scope of Matroid Minors Project

Part III

(Work in) Progress



Theorem (Mayhew, Whittle, vZ 2010+).

Rota's Conjecture for $\text{GF}(5)$ is implied by the Bounded Canopy Conjecture.

Guaranteed minors

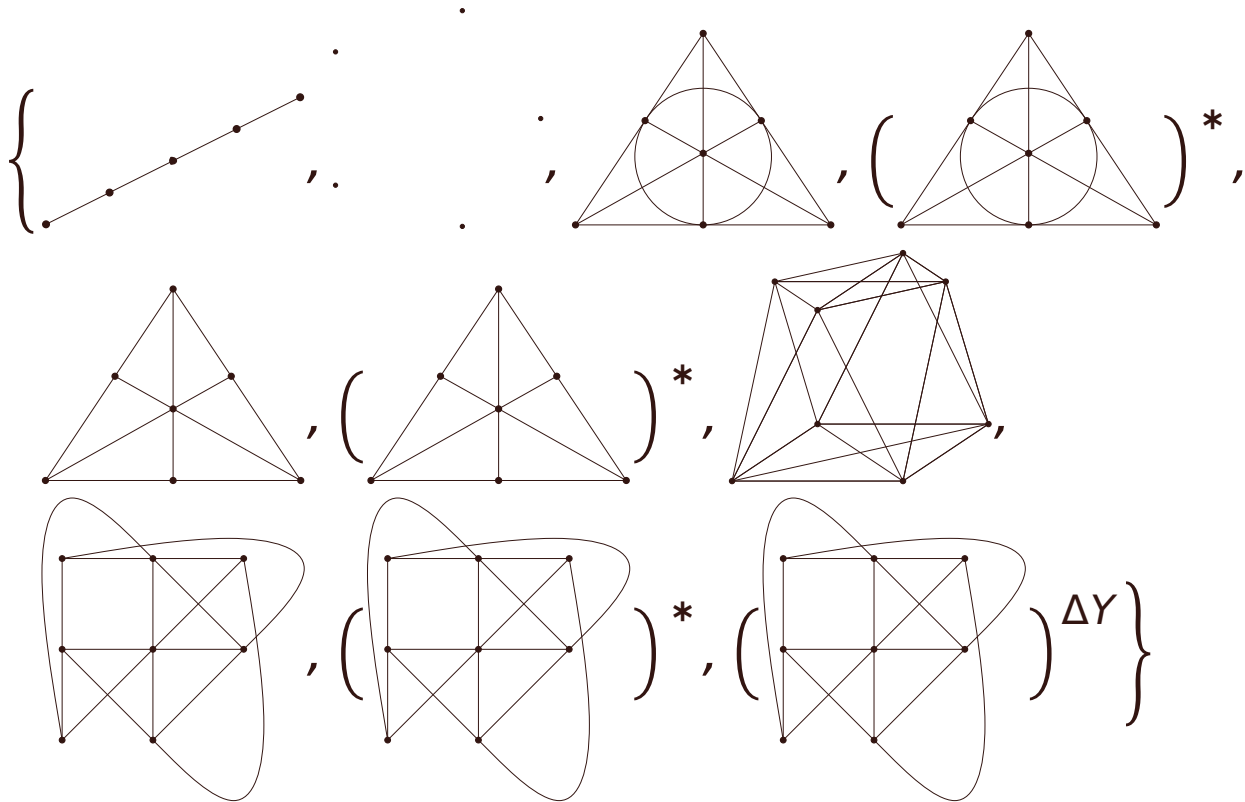
Definition.

M is *near-regular* if representable over $\text{GF}(3)$, $\text{GF}(4)$, $\text{GF}(5)$, \dots

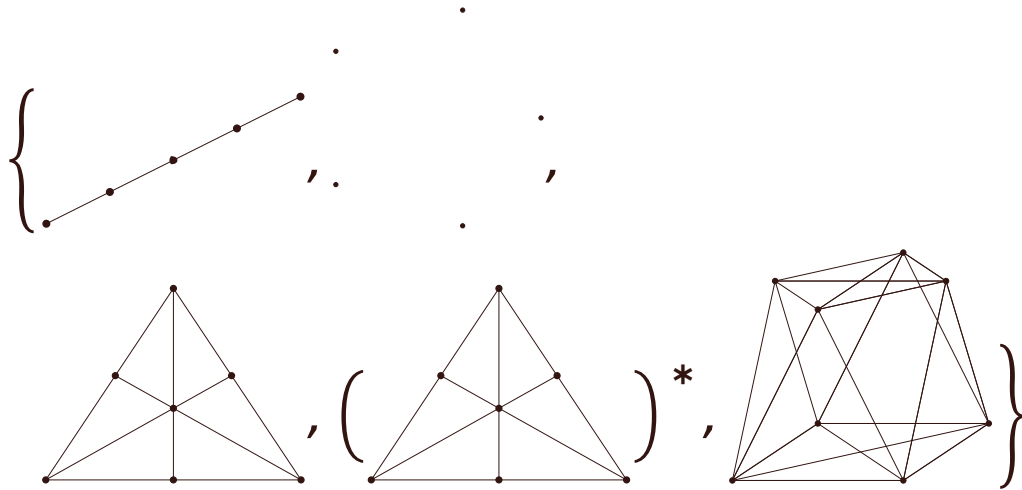
Theorem (Hall, Mayhew, vZ 2011).

Exactly 10 excluded minors for near-regular matroids

namely



Guaranteed minors



Ongoing work

- A *fan lemma*
- $\{U_{2,5}, U_{3,5}\}$ -fragile. Write-up phase.
- $\{F_7^-, (F_7^-)^*\}$ -fragile. Major work needed.
- $\{F_7^-, (F_7^-)^*, P_8\}$ -fragile with P_8 . Guessed the structure, need computer-aided check.

Truemper graphs

Theorem (Truemper 1992).

$\{F_7, F_7^*\}$ -fragile matroids are ΔY -reducible.

Truemper graphs

Theorem (Truemper 1992).

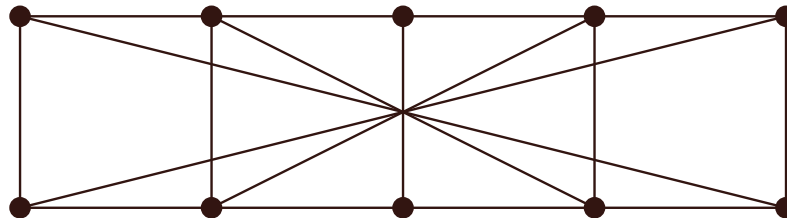
$\{F_7, F_7^*\}$ -fragile matroids are ΔY -reducible.

Need: explicit structure. Major tool:

Theorem (Mayhew, Whittle, vZ 2010+).

Equivalent are:

- G has a vital linkage of order 2;
- G has a spanning linkage of order 2 with no XX linkage minor;
- G is linkage minor of some





Slides, preprints at <http://www.cwi.nl/~zwam/>

The End