Fragility in matroid theory

Stefan van Zwam

Department of Mathematics
Princeton University

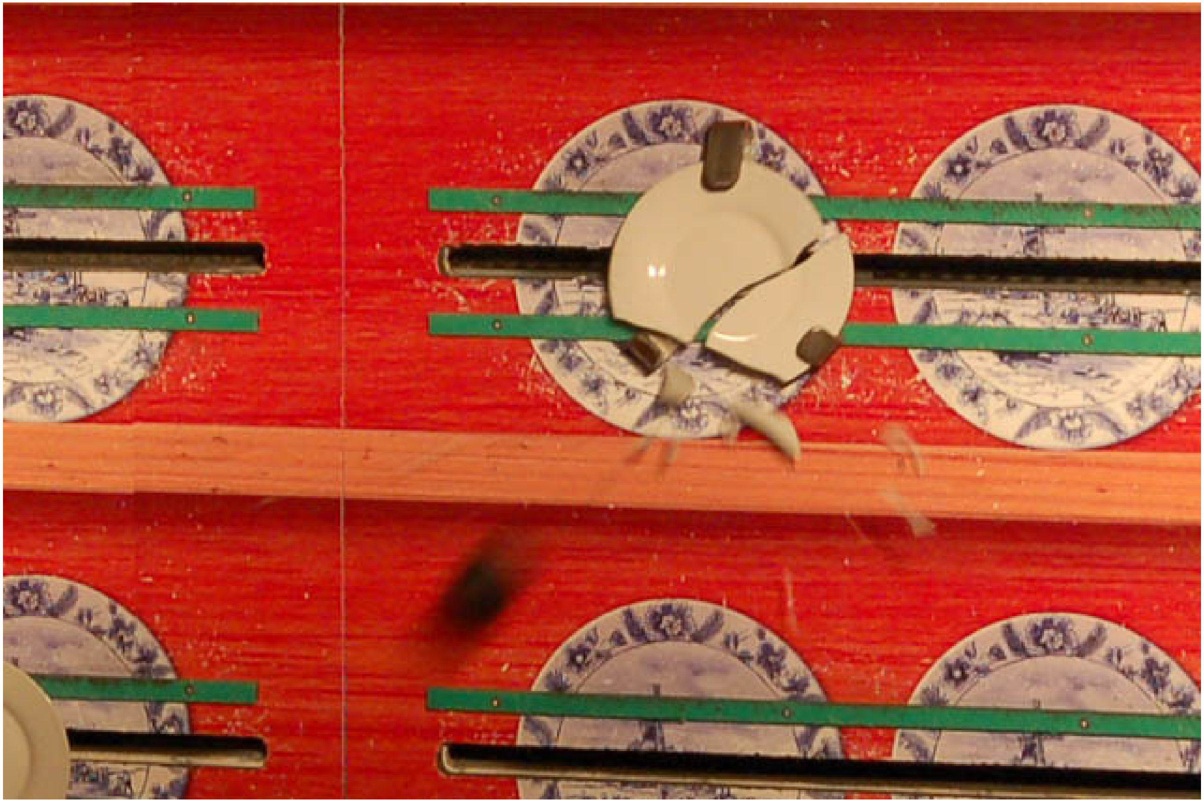
Based on joint and ongoing work with Carolyn Chun, Deborah Chun, Dillon Mayhew, and Geoff Whittle

Discrete Math Seminar, Columbia University, October 11, 2011

In today's presentation:

- Matroids and fragility
- Excluded minors
- (Work in) Progress

Part I
Matroids and fragility



What is a matroid?

ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.[1]

By Hassler Whitney.

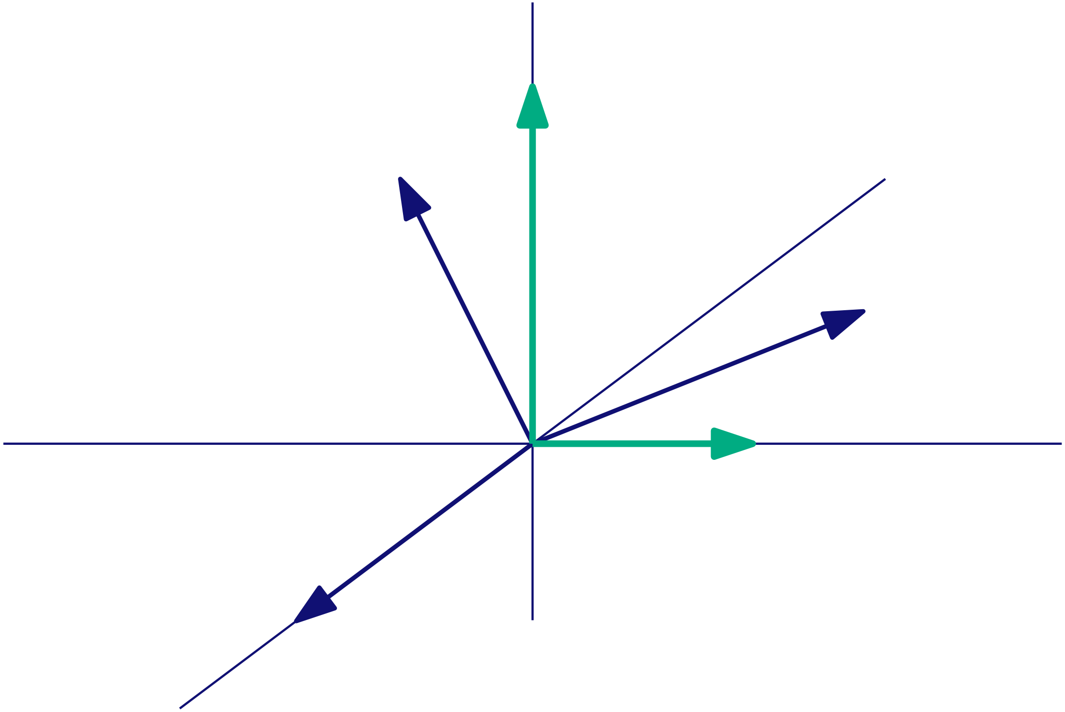
1. Introduction. Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

(a) Any subset of an independent set is independent.

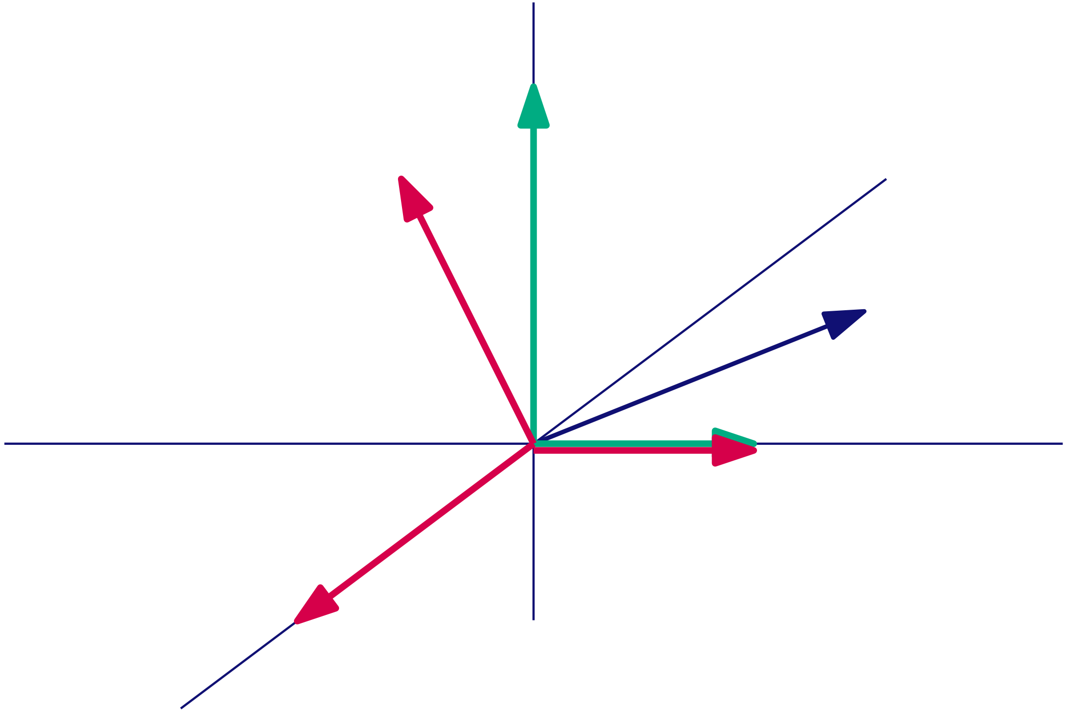
(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.

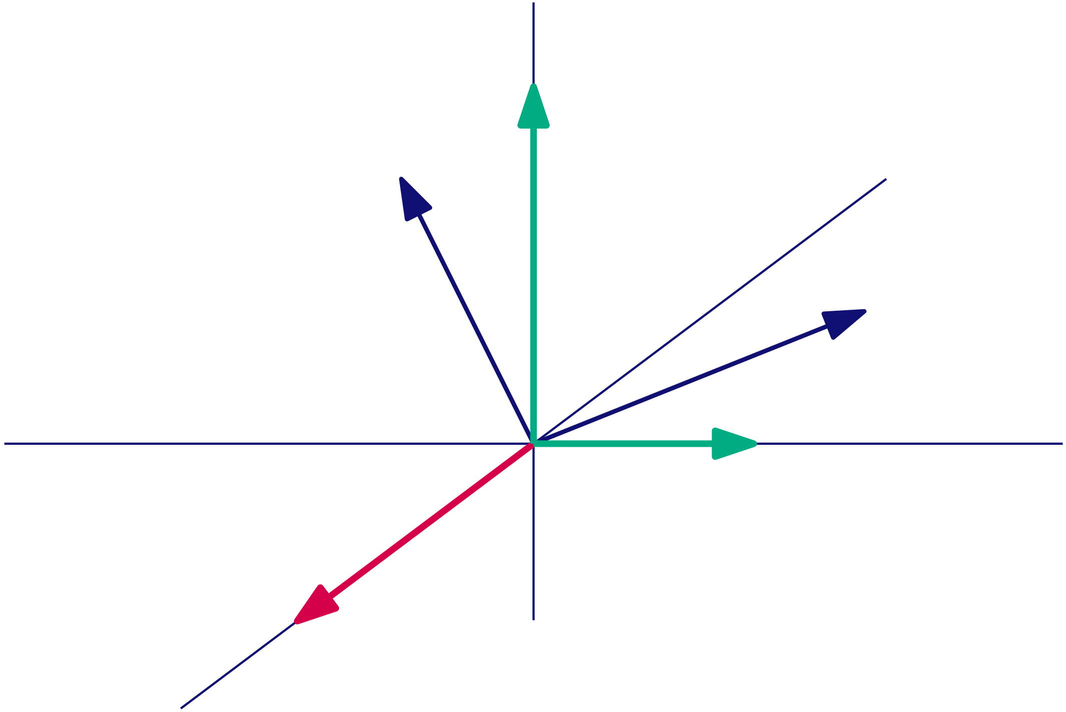
Linearly independent vectors in $\mathbb{R}^n$



Linearly independent vectors in $\mathbb{R}^n$



Linearly independent vectors in $\mathbb{R}^n$



Matroid axioms

Lemma. Given

E: finite set of vectors

$\mathcal{I}$: collection of linearly independent subsets

then

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

$$\exists e \in J - I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

Matroid axioms

Definition. Given

E: finite set

$\mathcal{I}$: collection of subsets

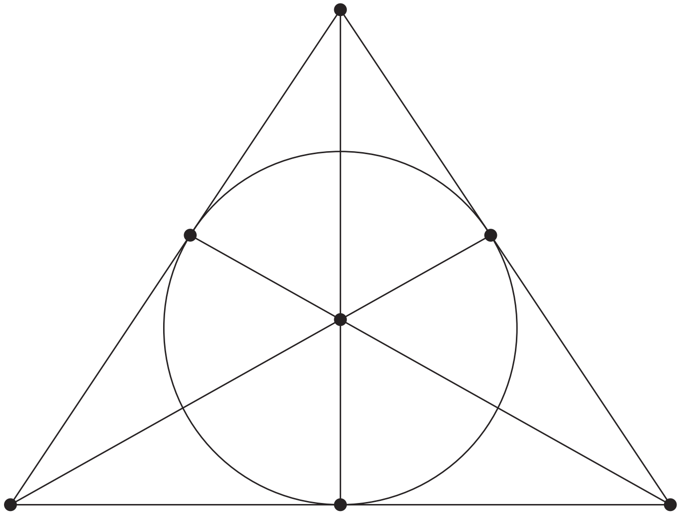
such that

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then

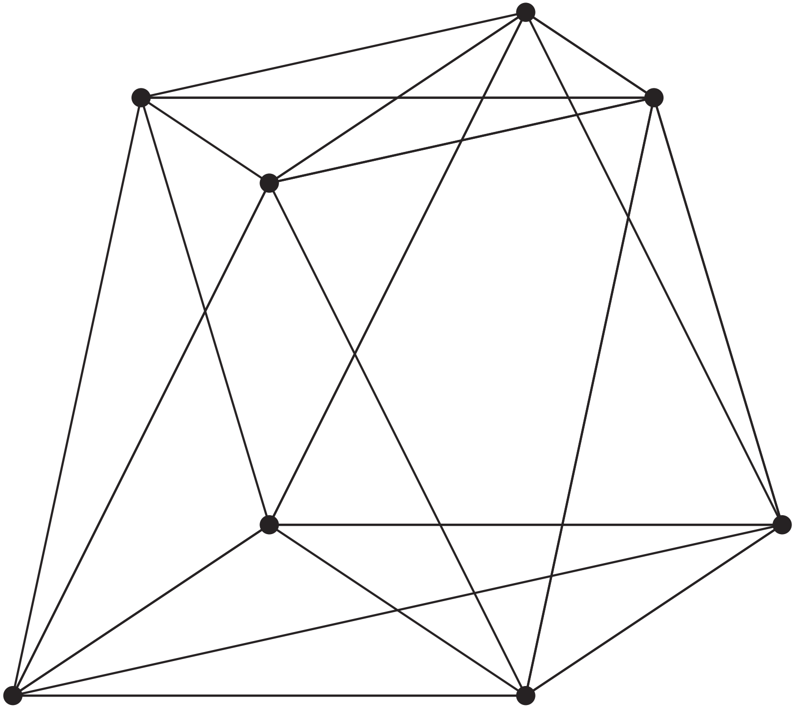
$$\exists e \in J - I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

Then $M = (E, \mathcal{I})$ is a **matroid**.

What is a matroid?



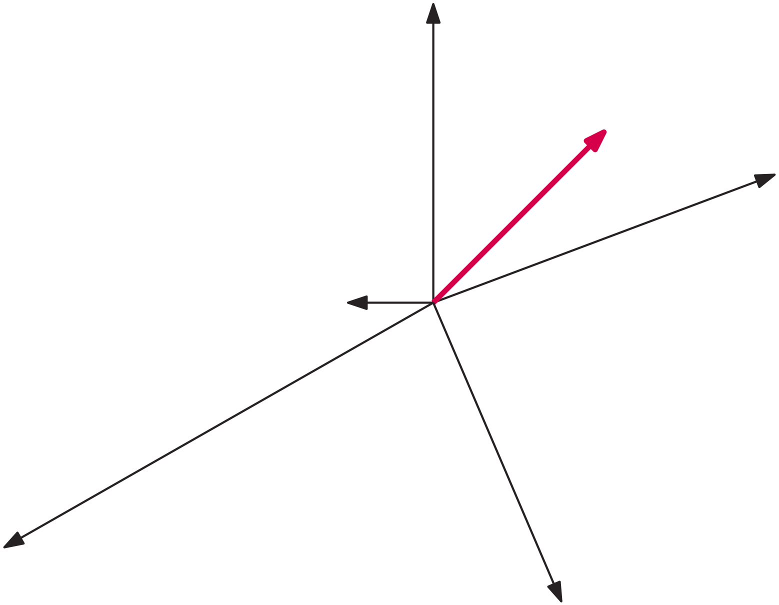
What is a matroid?



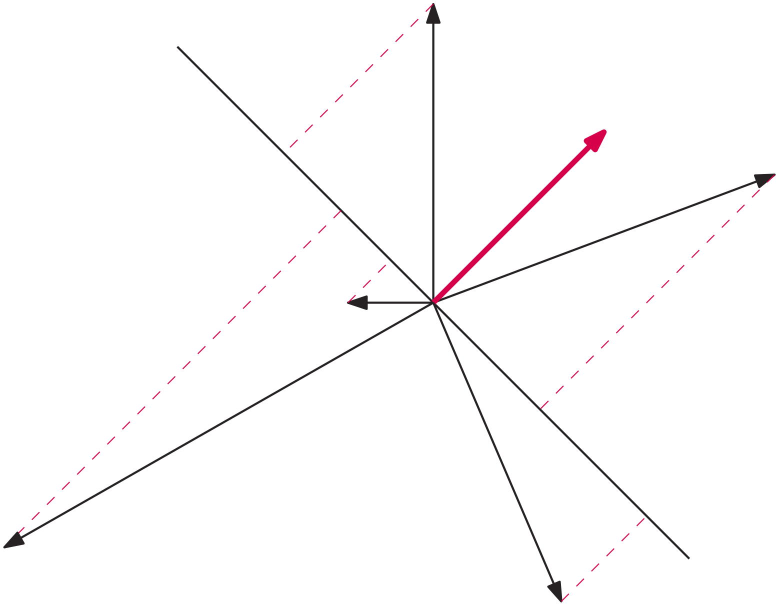
Minors

- *Deletion:* $M\backslash e := (E - \{e\}, \{I \in \mathcal{I} : e \notin I\})$
- *Contraction:* $M/e := (E - \{e\}, \{I : I \cup \{e\} \in \mathcal{I}\})$
- *Minors:* Obtained from sequence of such steps

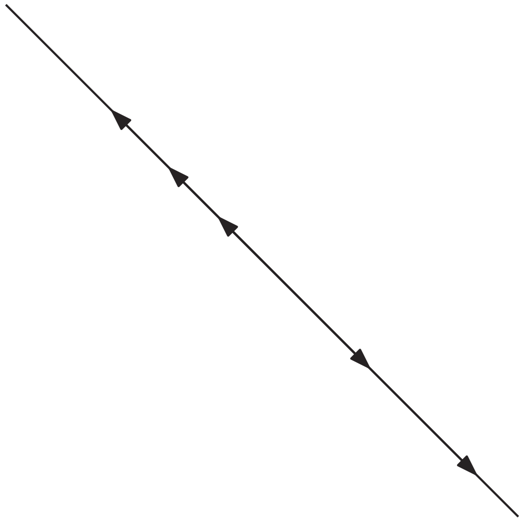
Contraction



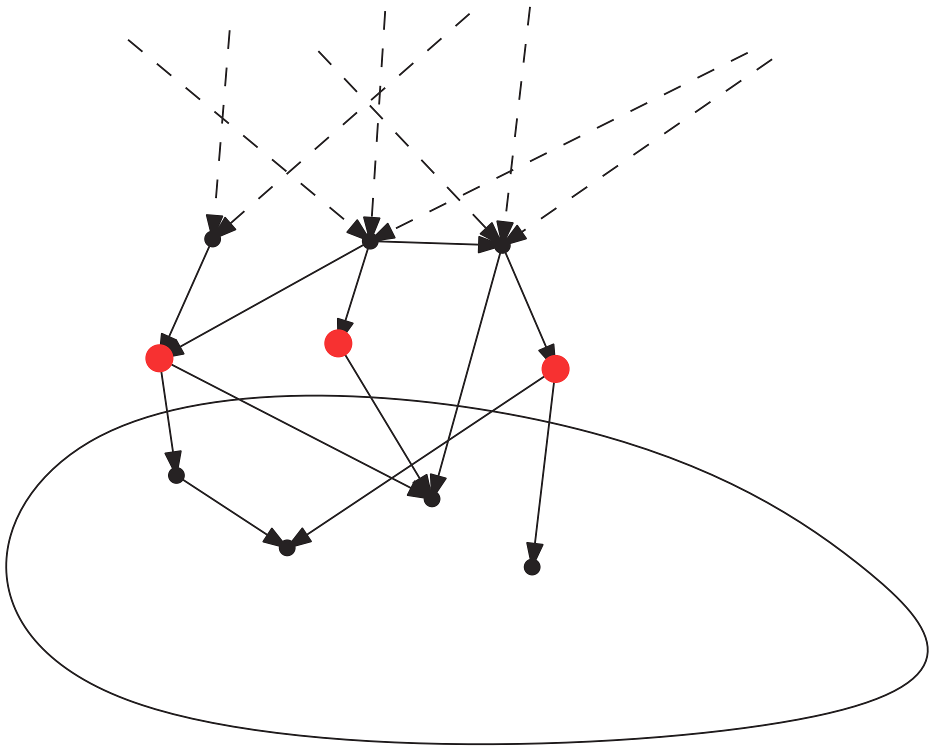
Contraction



Contraction



Minor order



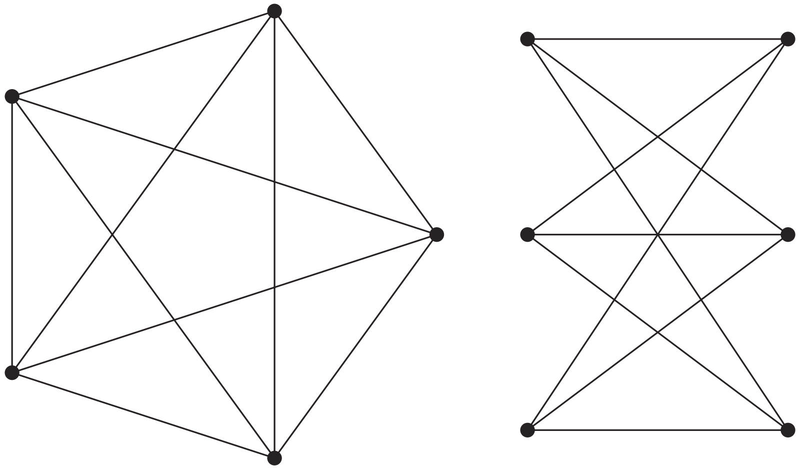
Excluded minors

Definition.
Matroid M is *excluded minor* for minor-closed class $\mathcal{C}$ if

- $M \notin \mathcal{C}$
- For all e: $M \backslash e$ **and** M/e in $\mathcal{C}$

Kuratowski’s Theorem

Theorem.
Exactly two excluded minors for planar graphs:



Fragility

First definition.

Matroid M is *almost-*$\mathcal{C}$ for minor-closed class $\mathcal{C}$ if

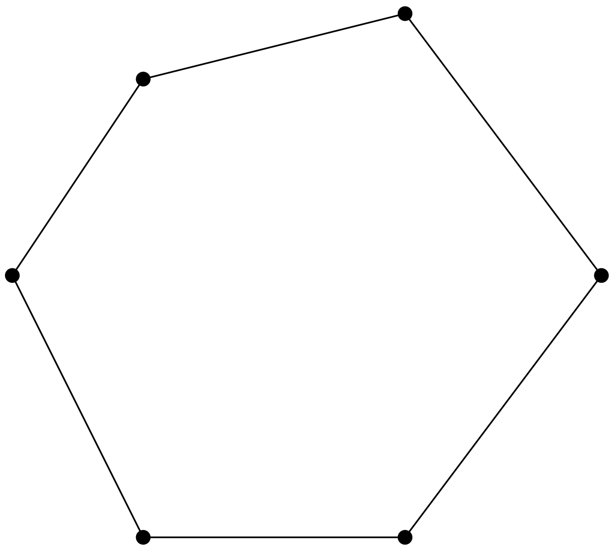
- $M \notin \mathcal{C}$
- For all e: $M \backslash e$ **or** M/e in $\mathcal{C}$

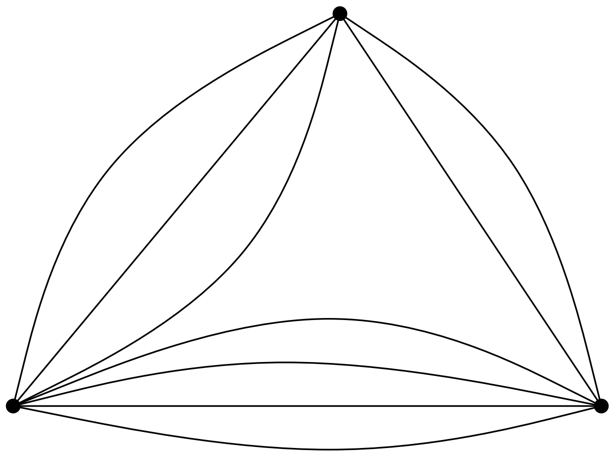
Example

Let G be an almost Δ-minor-free graph.

Example

Let G be an almost Δ-minor-free graph.





Example

Theorem (Gubser 1996).
Let G be a 3-connected almost-planar graph. Then G is a member of

$$\mathcal{B} \cup \mathcal{M} \cup \mathcal{H}_1 \cup \mathcal{H}_2.$$

Fragility

Definition.
Matroid M is $\mathcal{N}$-*fragile* for set of matroids $\mathcal{N}$ if

- For all e: **at most one** of $M \backslash e$ and M/e has a minor in $\mathcal{N}$.

Part II
Excluded minors

Wagner's Conjecture

Theorem (Robertson and Seymour, Graph Minors XX)
Let $\mathcal{C}$ be a minor-closed class of graphs. There is a finite number of excluded minors for $\mathcal{C}$.

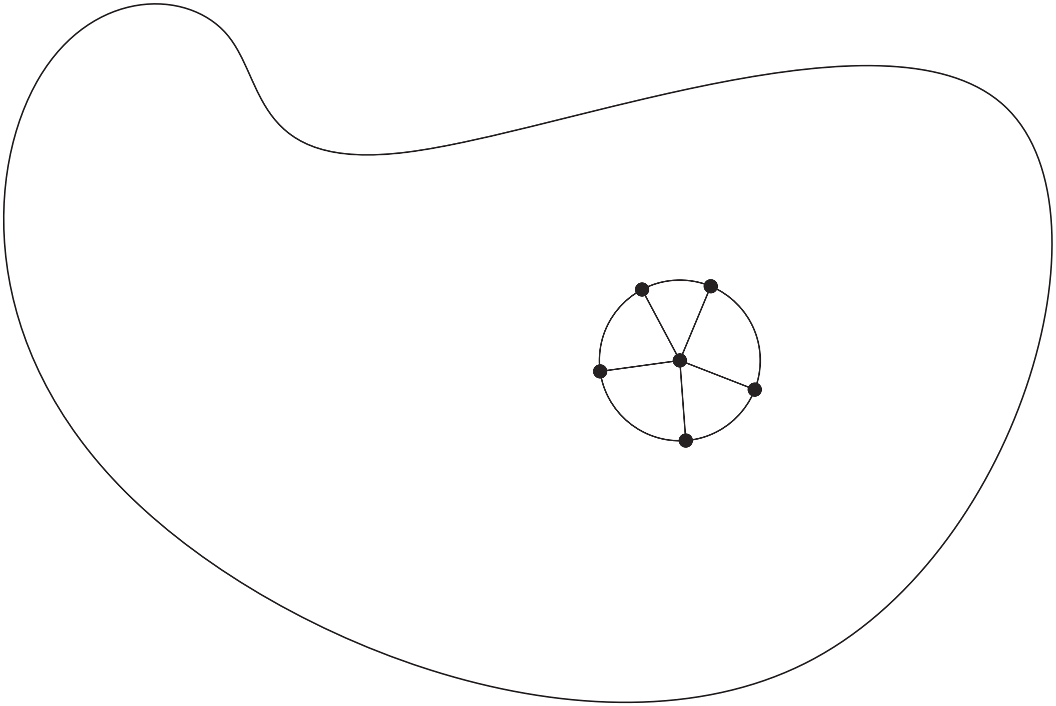
Wagner's Conjecture

Theorem (Robertson and Seymour, Graph Minors XX)
Let $\mathcal{C}$ be a minor-closed class of graphs. There is a finite number of excluded minors for $\mathcal{C}$.

Theorem (Robertson and Seymour)
There is a polynomial-time algorithm to test if $G \in \mathcal{C}$.

Irrelevant vertex

- Low tree width: dynamic programming
- High tree width: find irrelevant vertex

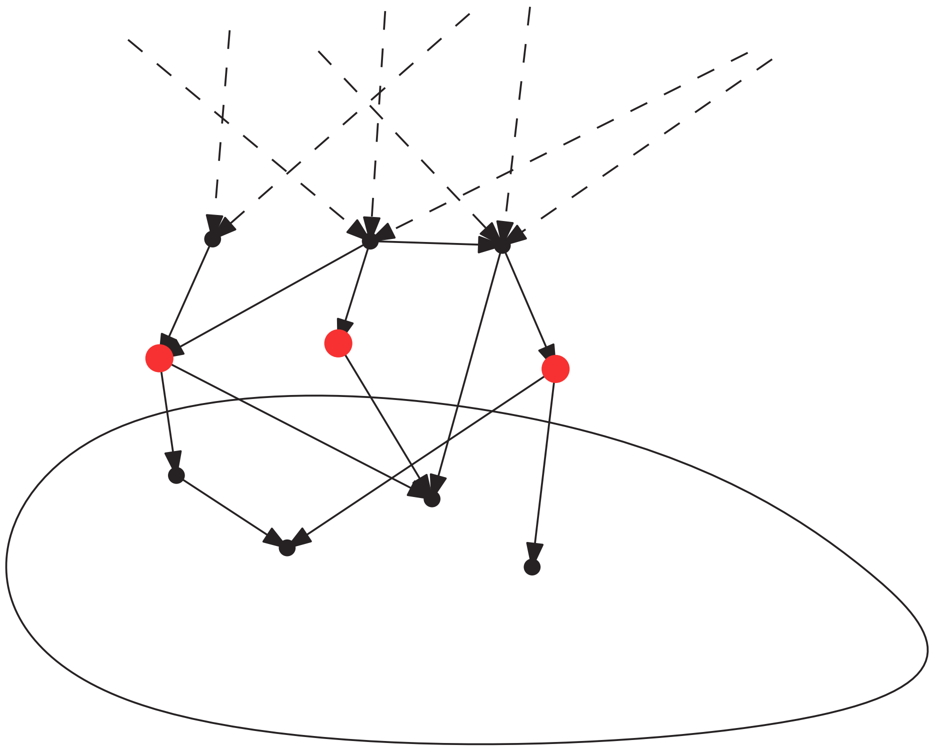


Matroid Minors Project

Bad news. Let $\mathcal{C}$ be all real-representable matroids.

Theorem (Mayhew, Newman, Whittle 2009).
Let M be a real-representable matroid. Then there is an excluded minor for $\mathcal{C}$ having M as minor.

Minor order



Matroid Minors Project

However. . .

Let $\mathcal{C}_q$ be all matroids representable over GF(q).

Conjecture (and work in progress by Geelen, Gerards, Whittle).
Let $\mathcal{C}$ be a minor-closed subclass of $\mathcal{C}_q$. There is a finite number of excluded minors for $\mathcal{C}$ in $\mathcal{C}_q$.

Matroid Minors Project

However...

Let $\mathcal{C}_q$ be all matroids representable over GF(q).

Conjecture (and work in progress by Geelen, Gerards, Whittle).
Let $\mathcal{C}$ be a minor-closed subclass of $\mathcal{C}_q$. There is a finite number of excluded minors for $\mathcal{C}$ in $\mathcal{C}_q$.

Conjecture (and work in progress by Geelen, Gerards, Whittle).
There is a polynomial-time algorithm to test if $M \in \mathcal{C}$, if representation given.

Flexible element

- Low branch width: dynamic programming.
- High branch width: find *flexible element*: $M\backslash e$ and M/e both have N-minor.

Flexible element

- Low branch width: dynamic programming.
- High branch width: find *flexible element*: $M \backslash e$ and M/e both have N-minor.
- ... in other words: M is not N-*fragile*

Bounded Canopy Conjecture

Conjecture (Geelen, Gerards, Whittle 2006).
$\exists k = k(N, \mathbb{F})$:
If M is $\mathbb{F}$-representable, strictly N-fragile then

$$\mathrm{bw}(M) \leq k$$

Rota's Conjecture

Let $\mathcal{C}_q$ be all matroids representable over GF(q).

Conjecture (Rota 1971).
There is a finite number of excluded minors for $\mathcal{C}_q$.

Rota's Conjecture

Let $\mathcal{C}_q$ be all matroids representable over GF(q).

Conjecture (Rota 1971).
There is a finite number of excluded minors for $\mathcal{C}_q$.

- Matroid Theorists' Holy Grail
- Proven for $q \leq 4$
- Outside scope of Matroid Minors Project

Part III
(Work in) Progress

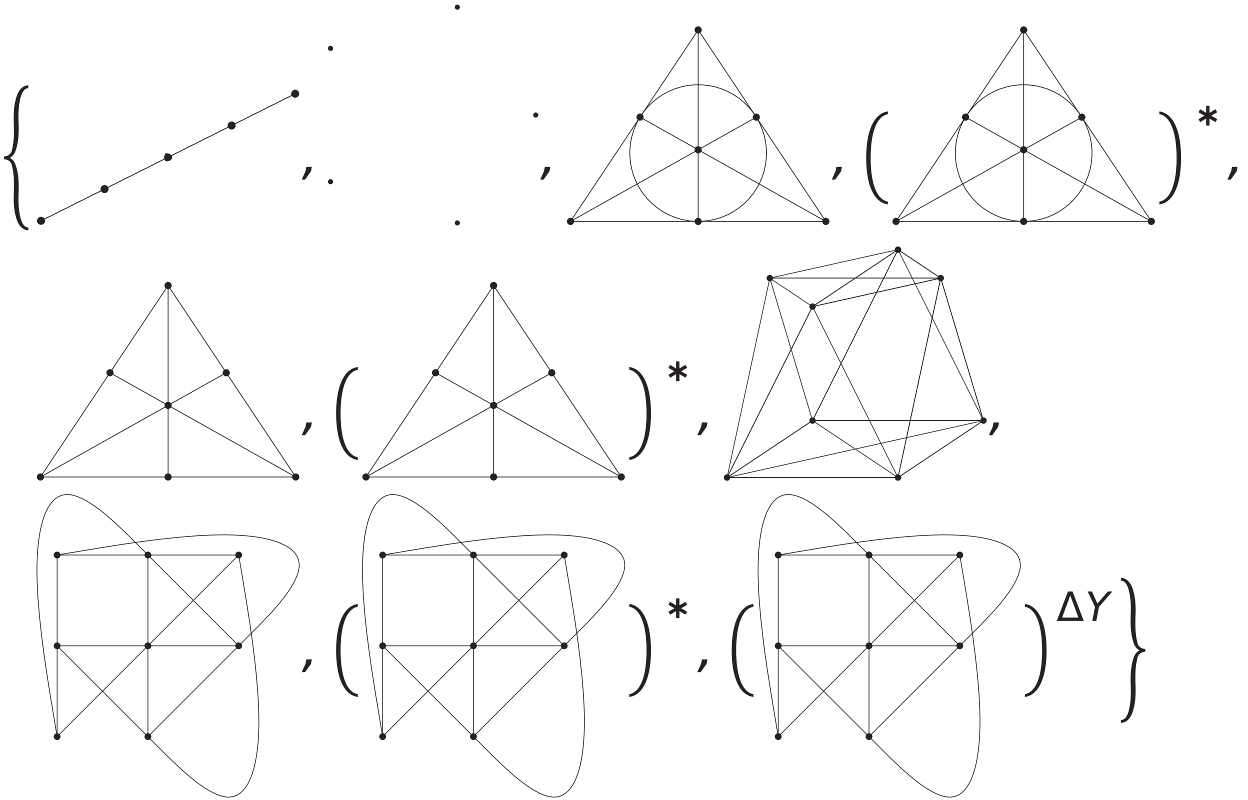
Theorem (Mayhew, Whittle, vZ 2010+).
Rota's Conjecture for GF(5) is implied by the Bounded Canopy Conjecture.

Guaranteed minors

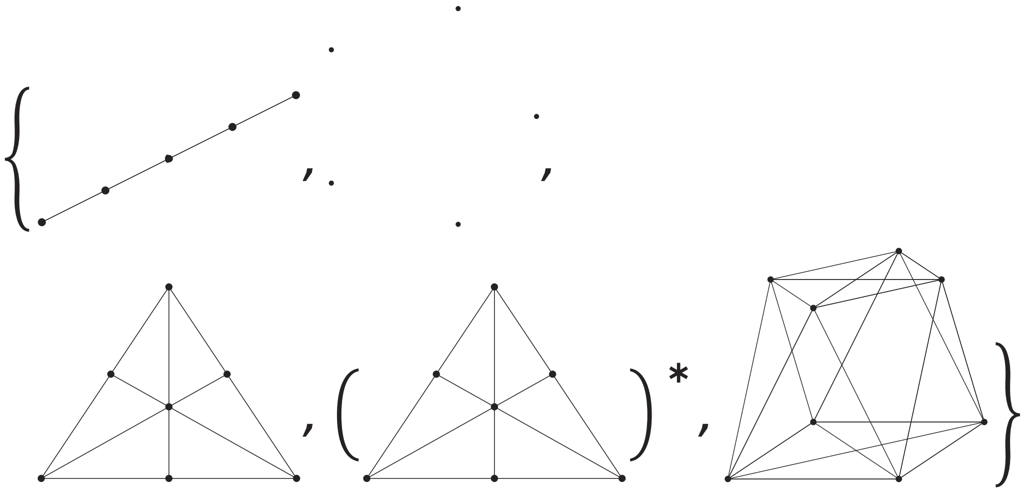
Definition.
M is *near-regular* if representable over $\mathrm{GF}(3), \mathrm{GF}(4), \mathrm{GF}(5), \ldots$

Theorem (Hall, Mayhew, vZ 2011).
Exactly 10 excluded minors for near-regular matroids

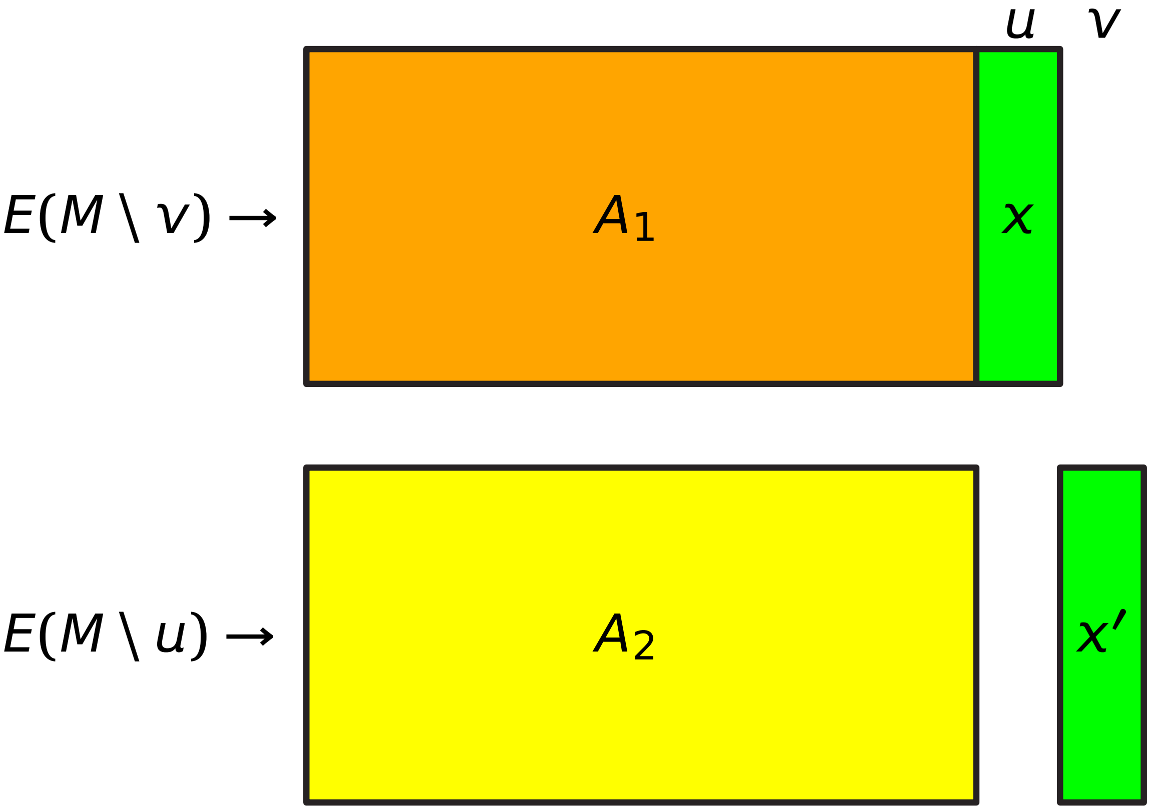
namely



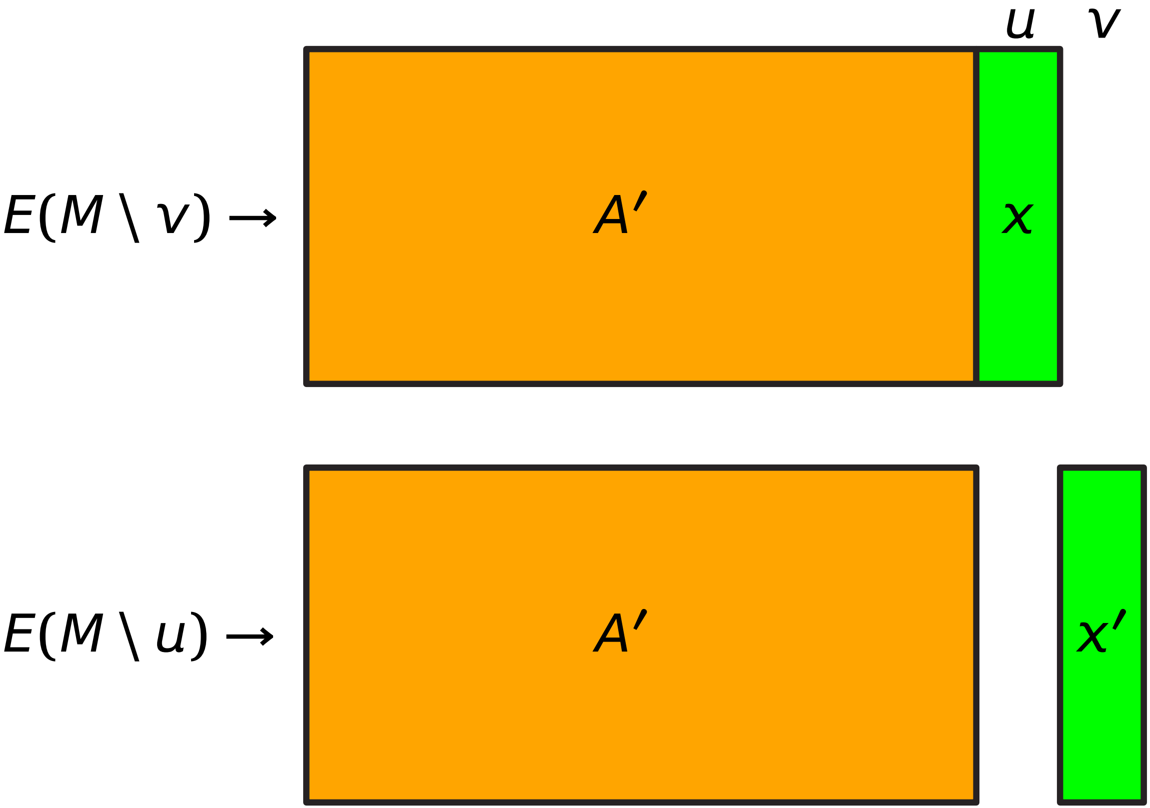
Guaranteed minors



Sketch of proof

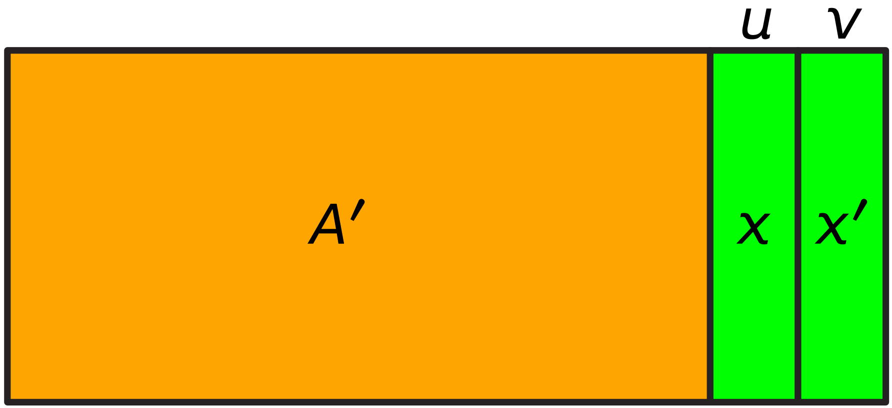


Sketch of proof



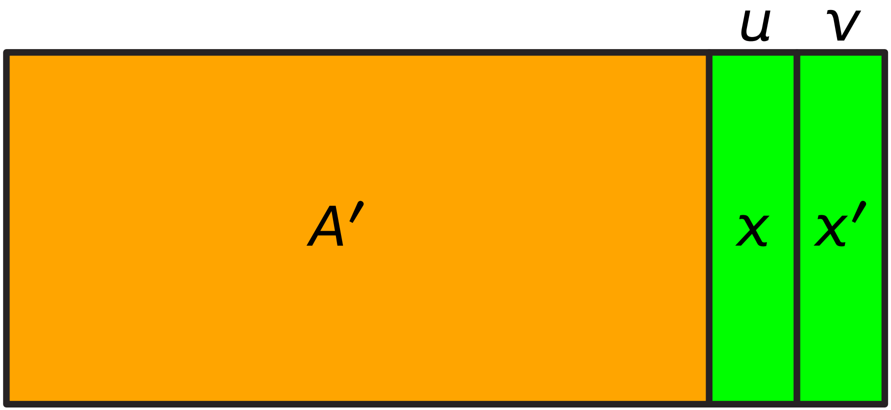
Sketch of proof

$A =$



Sketch of proof

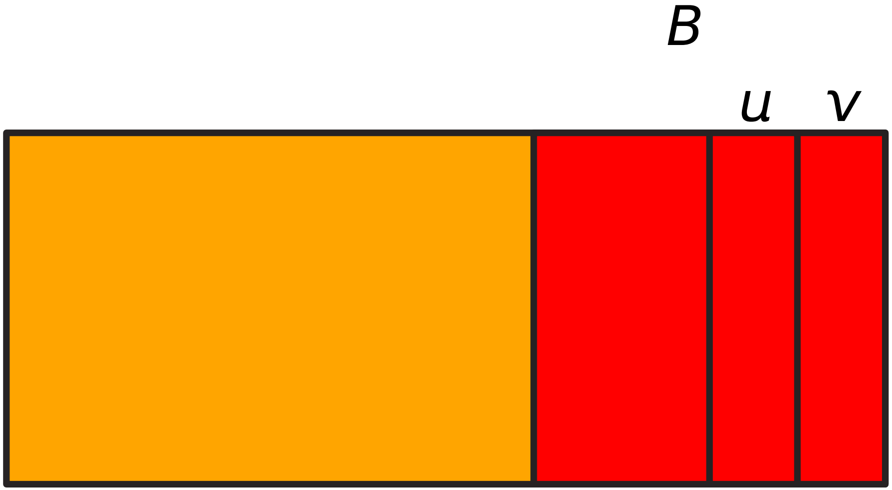
$A =$



$$M \neq M[A]$$

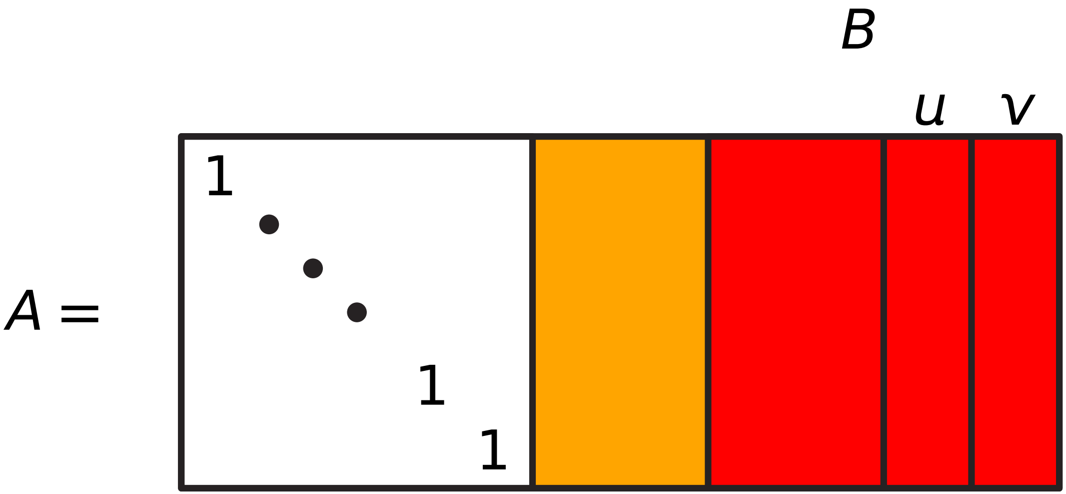
Sketch of proof

$A =$



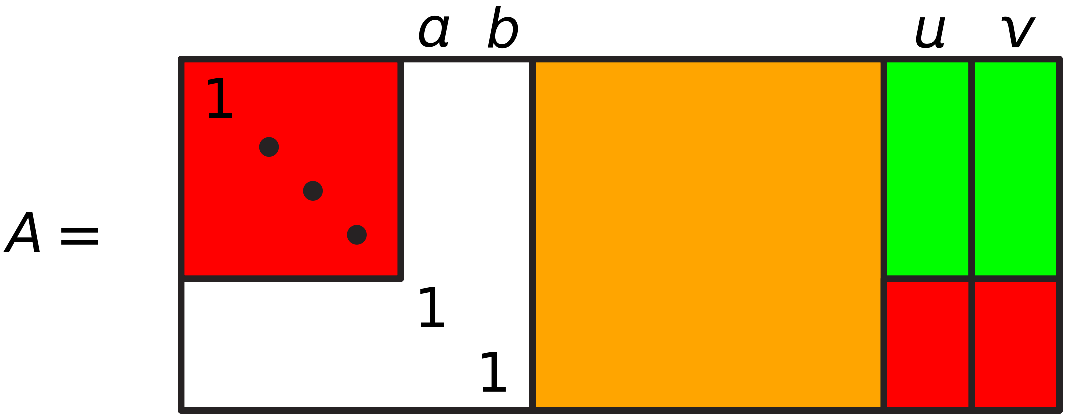
B basis in one of $M, M[A]$

Sketch of proof



B basis in one of $M, M[A]$

Sketch of proof



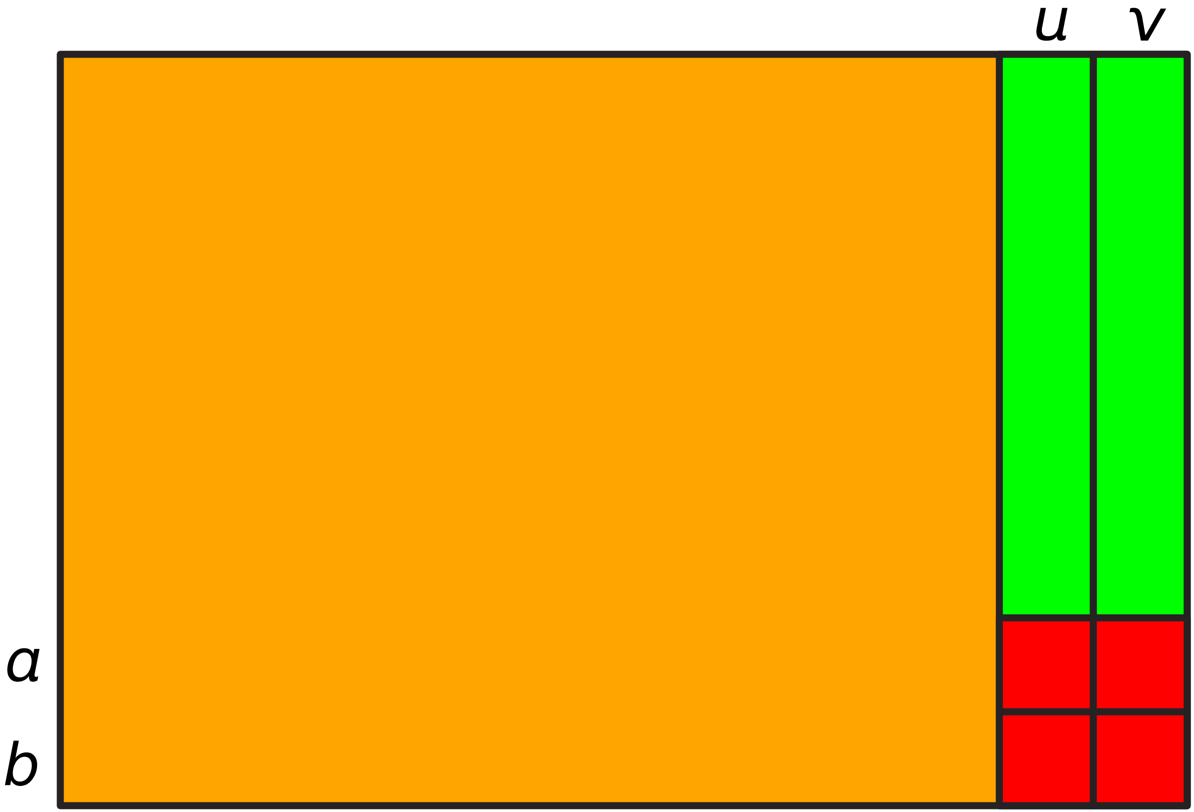
B basis in one of $M, M[A]$

Sketch of proof

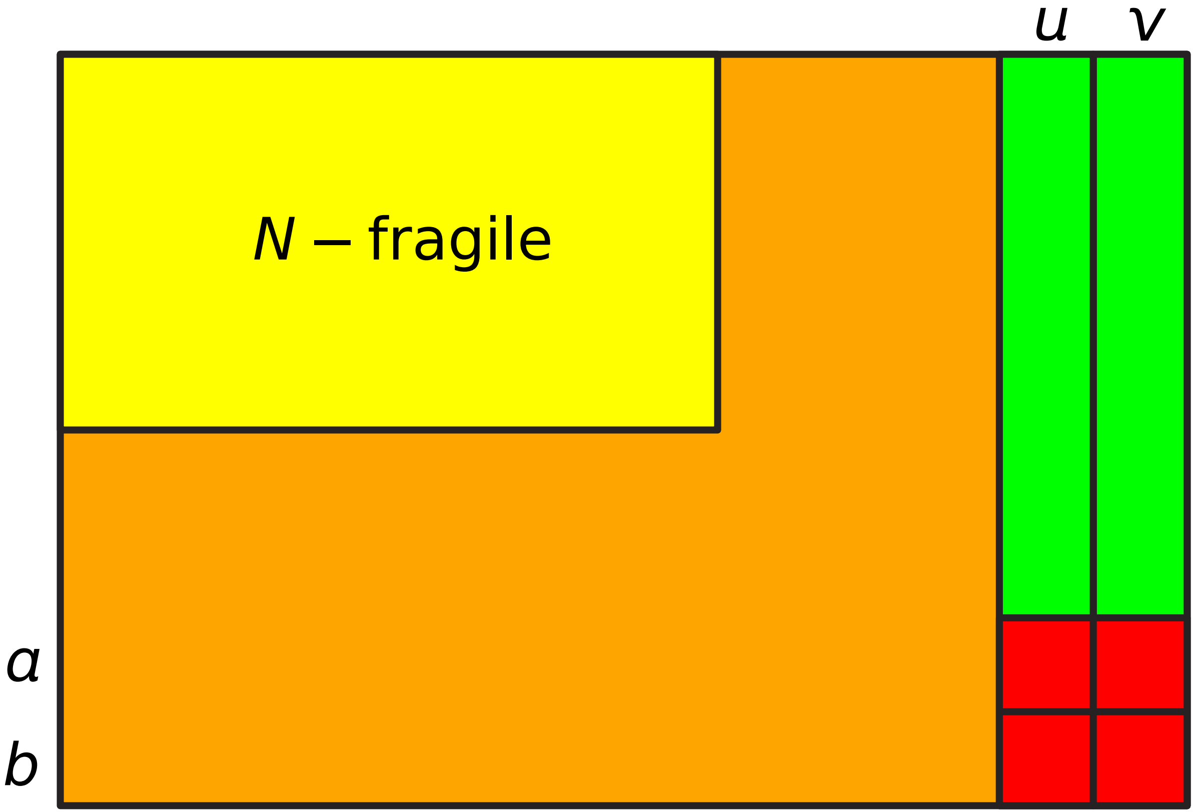
Essential properties of M and A:

- “Bad” submatrix
- Guaranteed minor N
- 3-connected

Where’s *N*?

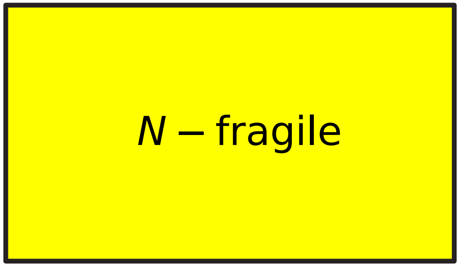


Where's N?



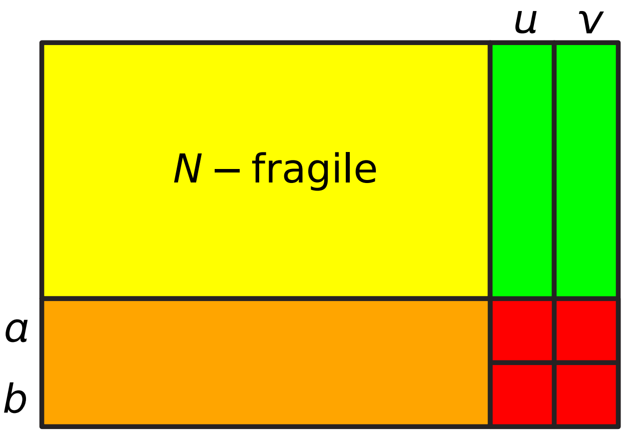
Putting it together

- N-fragile minor
- Add back a, b, u, v for “bad” submatrix
- Repair connectivity



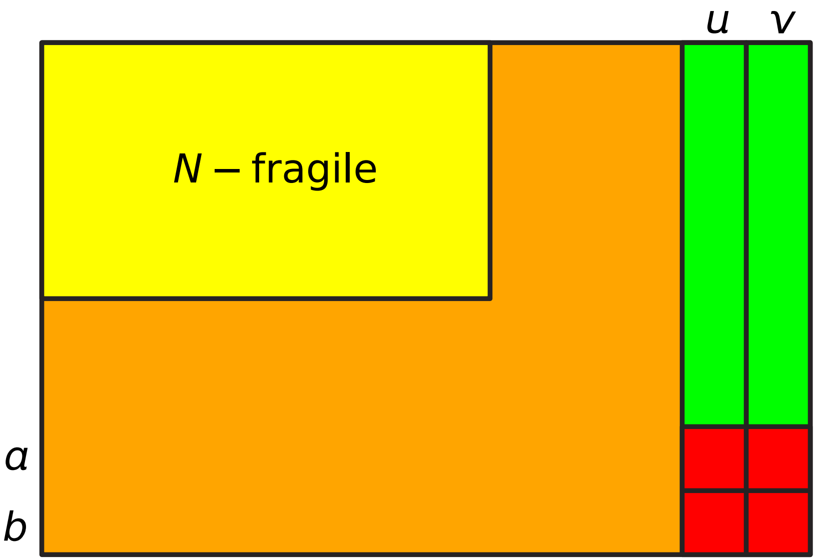
Putting it together

- N-fragile minor
- Add back a, b, u, v for "bad" submatrix
- Repair connectivity



Putting it together

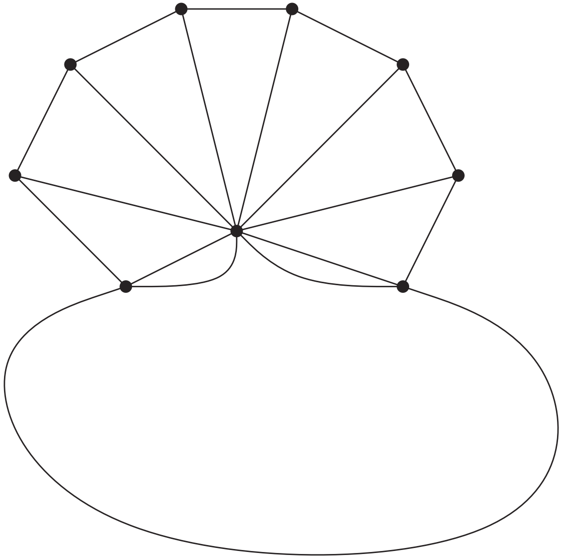
- N-fragile minor
- Add back a, b, u, v for “bad” submatrix
- Repair connectivity

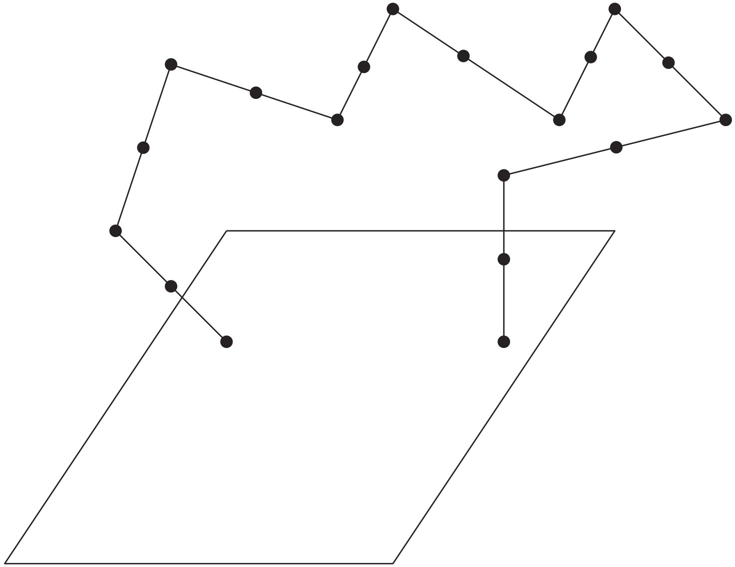


Ongoing work

- A *fan lemma*
- $\{U_{2,5}, U_{3,5}\}$-fragile. Write-up phase.
- $\{F_7^-, (F_7^-)^*\}$-fragile. Major work needed.
- $\{F_7^-, (F_7^-)^*, P_8\}$-fragile with P_8. Guessed the structure, need computer-aided check.

Fans





$\{F_7^-, (F_7^-)^*\}$-fragile matroids

Significant subclass: $U_{2,4}$-fragile matroids.

Theorem (Oxley 1992).
Every $U_{2,4}$-fragile matroid is a circuit-hyperplane *relaxation* of a binary matroid.

$\{F_7^-, (F_7^-)^*\}$-fragile matroids

Significant subclass: $U_{2,4}$-fragile matroids.

Theorem (Oxley 1992).
Every $U_{2,4}$-fragile matroid is a circuit-hyperplane *re-laxation* of a binary matroid.

- Huge class, no bound on branch width

$\{F_7^-, (F_7^-)^*\}$-fragile matroids

Significant subclass: $U_{2,4}$-fragile matroids.

Theorem (Oxley 1992).
Every $U_{2,4}$-fragile matroid is a circuit-hyperplane *relaxation* of a binary matroid.

- Huge class, no bound on branch width
- If representable, then one of the following:
  - Generalized whirl
  - Relaxation of $\{F_7, F_7^*\}$-fragile, binary matroid

Truemper graphs

Theorem (Truemper 1992).
$\{F_7, F_7^*\}$-fragile matroids are ΔY-reducible.

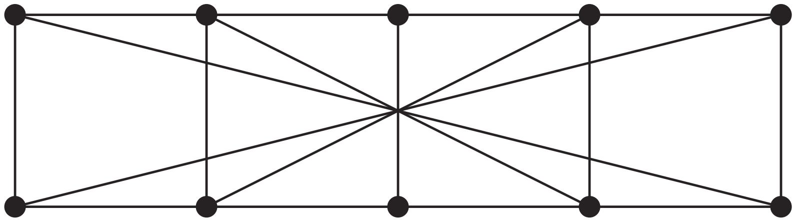
Truemper graphs

Theorem (Truemper 1992).
$\{F_7, F_7^*\}$-fragile matroids are ΔY-reducible.

Need: explicit structure. Major tool:

Theorem (Mayhew, Whittle, vZ 2010+).
Equivalent are:

- G has a vital linkage of order 2;
- G has a spanning linkage of order 2 with no XX linkage minor;
- G is linkage minor of some



Slides, preprints at
http://www.math.princeton.edu/~svanzwam/

The End