The structure of graphs with a vital linkage of order 2

Stefan van Zwam

Department of Mathematics
Princeton University

Based on joint and ongoing work with Carolyn Chun, Deborah Chun, Dillon Mayhew, and Geoff Whittle

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In today's presentation:

- Matroids and Rota's Conjecture
- Fragility
- Linkages

Part I
Matroids and Rota's Conjecture

What is a matroid?

ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.[1]

By Hassler Whitney.

1. Introduction. Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

(a) Any subset of an independent set is independent.

(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental question of completely characterizing systems which represent matrices is left unsolved. In place of the columns of a matrix we may equally well consider points or vectors in a Euclidean space, or polynomials, etc.

Linearly independent vectors in $\mathbb{R}^n$

- Finite set of vectors:

$$E = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \end{bmatrix} \right\}$$

- *Circuits:* minimal dependent subsets.

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 6 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Matroid axioms

Lemma. Given

E: finite set of vectors

$\mathcal{C}$: collection of minimal lin. dependent subsets

then

- $\emptyset \notin \mathcal{C}$
- $C, D \in \mathcal{C}$ and $C \subseteq D$ then $C = D$
- $C, D \in \mathcal{C}$ and $e \in C \cap D$ then have $F \subseteq C \cup D - e$ and $F \in \mathcal{C}$

Matroid axioms

Definition. Given

E: finite set

$\mathcal{C}$: collection of subsets

such that

- $\emptyset \notin \mathcal{C}$
- $C, D \in \mathcal{C}$ and $C \subseteq D$ then $C = D$
- $C, D \in \mathcal{C}$ and $e \in C \cap D$ then have $F \subseteq C \cup D - e$ and $F \in \mathcal{C}$

Then $M = (E, \mathcal{C})$ is a **matroid**.

Representability

Representation of M over field $\mathbb{F}$: circuit-preserving map

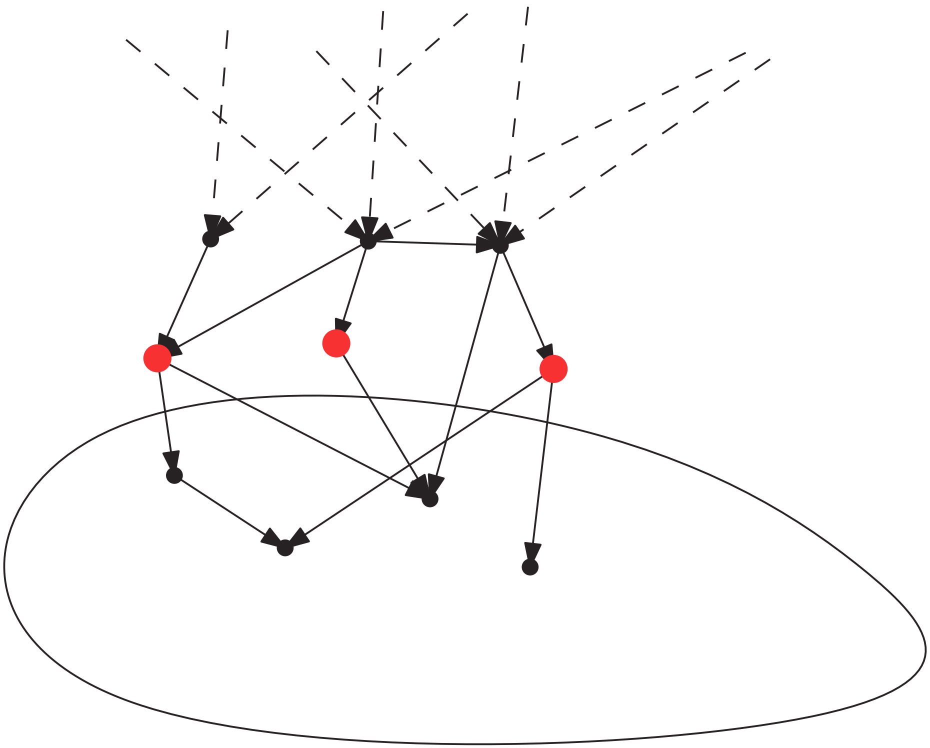
$$A : E(M) \to \mathbb{F}^r$$

for some $r > 0$.

Minors

- *Deletion:* $M \backslash e := (E - \{e\}, \{C \in \mathcal{C} : e \notin C\})$
- *Contraction:* $M/e := (E - \{e\}, \{C : stuff\})$
- *Minor:* Obtained from sequence of such steps

Minor order



Excluded minors

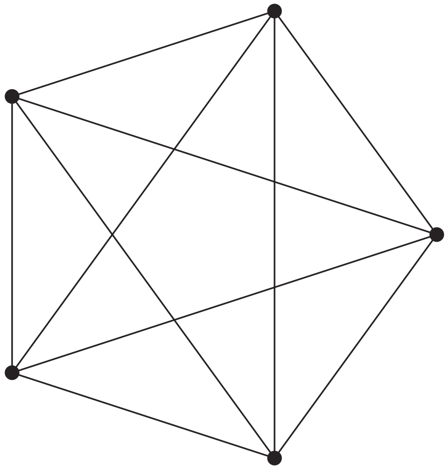
Definition.
Matroid M is *excluded minor* for minor-closed class $\mathcal{X}$ if

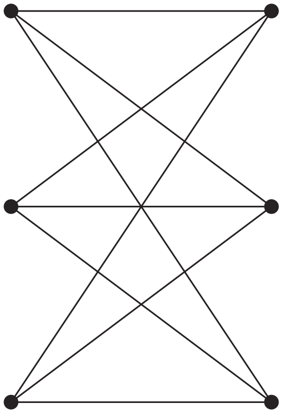
- $M \notin \mathcal{X}$
- For all e: $M \backslash e$ **and** M/e in $\mathcal{X}$

Kuratowski's Theorem

Theorem.

Exactly two excluded minors for planar graphs:





Rota's Conjecture

Let $\mathcal{X}_q$ be all matroids representable over GF(q).

Conjecture (Rota 1971).
There is a finite number of excluded minors for $\mathcal{X}_q$.

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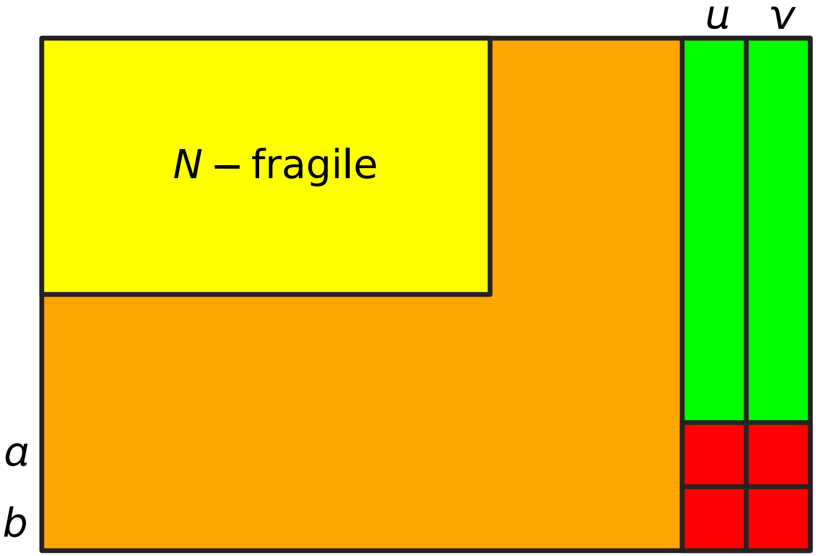
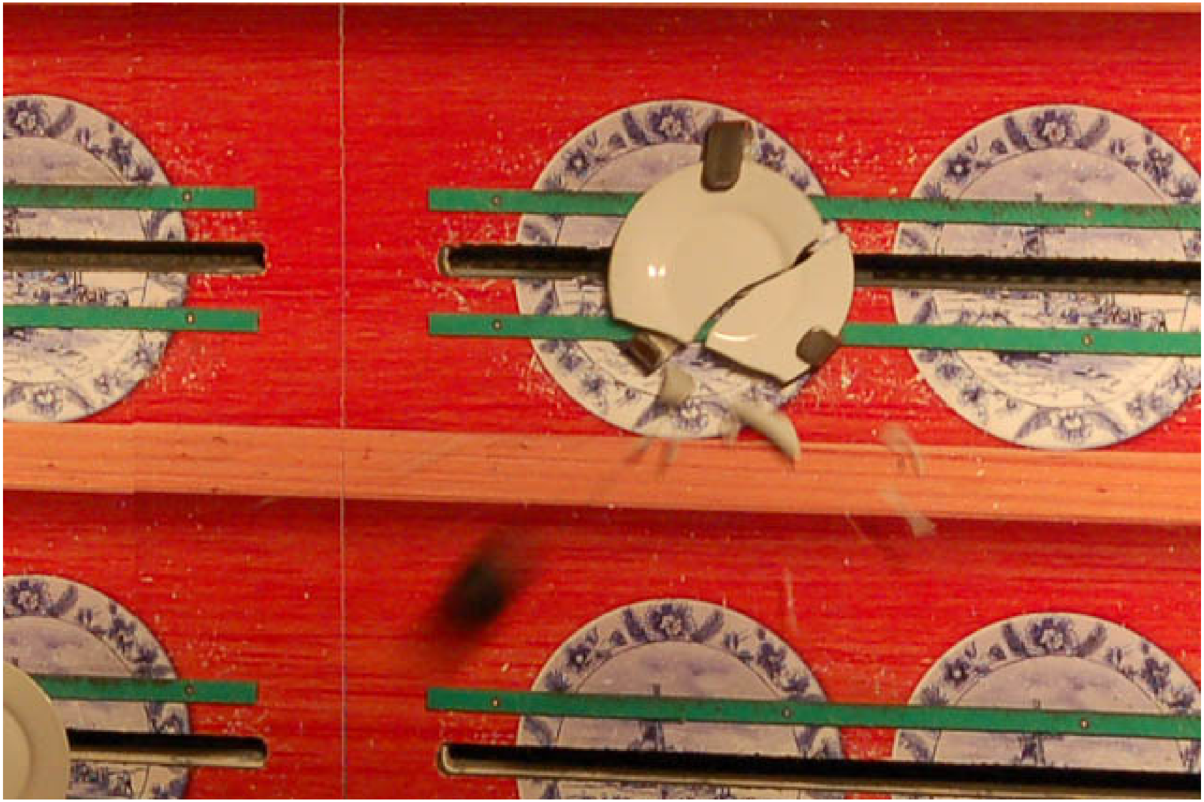
- Matroid Theorists' Holy Grail
- Proven for $q \leq 4$

Rota's Conjecture for GF(5)

- (Mayhew, Whittle, vZ): follows from Matroid Minors Structure Theory
- Goal: find them explicitly
- (Royle, Mayhew): computation: > 500
- Know lots about structure:



Part II
Fragility



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Fragility

First definition.
Matroid M is *almost-*$\mathcal{X}$ for minor-closed class $\mathcal{X}$ if

- $M \notin \mathcal{X}$
- For all e: $M\backslash e$ **or** M/e in $\mathcal{X}$

Example

Theorem (Gubser 1996).
Let G be a 3-connected almost-planar graph. Then G is a member of

$$\mathcal{B} \cup \mathcal{M} \cup \mathcal{H}_1 \cup \mathcal{H}_2.$$

Fragility

Definition.
Matroid M is $\mathcal{N}$-*fragile* for set of matroids $\mathcal{N}$ if

- For all e: **at most one** of $M \backslash e$ and M/e has a minor in $\mathcal{N}$.

A fragility question

Problem.
Characterize the binary $\{F_7, F_7^*\}$-fragile matroids.

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Theorem (Truemper 1992).
$\{F_7, F_7^*\}$-fragile matroids are ΔY-reducible.

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Need: explicit structure.

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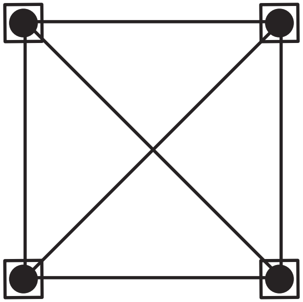
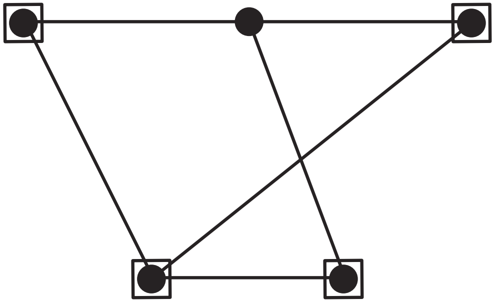
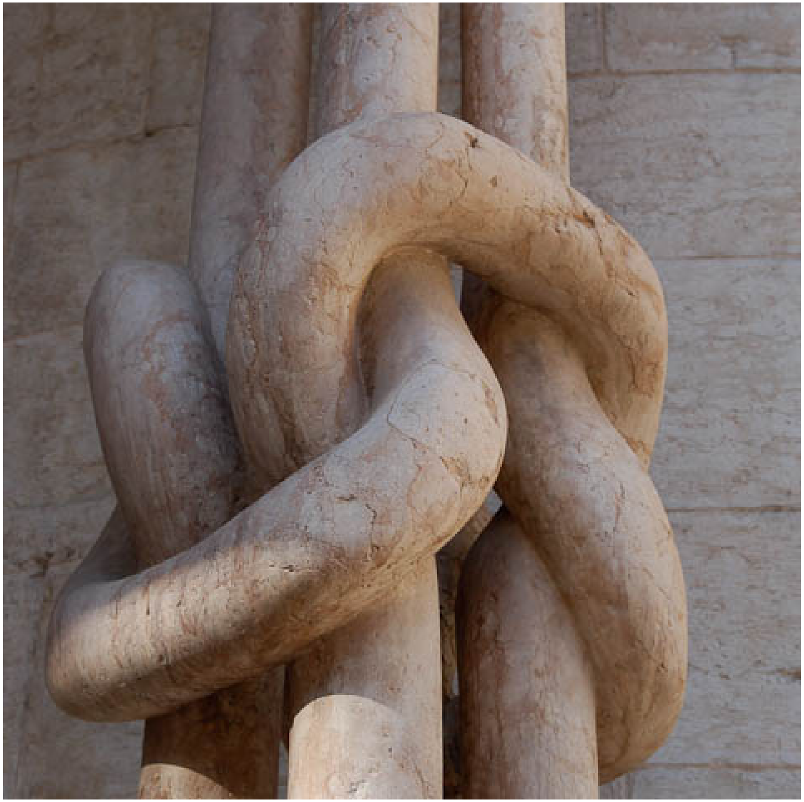


(a)



(b)

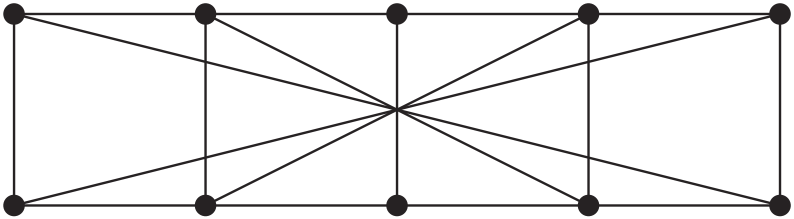
Part III
Linkages



Truemper graphs

Theorem (Mayhew, Whittle, vZ 2010+).
Equivalent are:

- *G* has vital linkage of order 2;
- *G* has chordless spanning linkage of order 2 with no *XX* linkage minor;
- *G* is linkage minor of some



Slides, preprints at
http://www.math.princeton.edu/~svanzwam/

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