Beyond Total Unimodularity

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The plan:

- Matroid Representations, Whittle’s Classes
- Basis Counting
- Decomposition
- Excluded Minors

Part I
Matroid Representations, Whittle's Classes

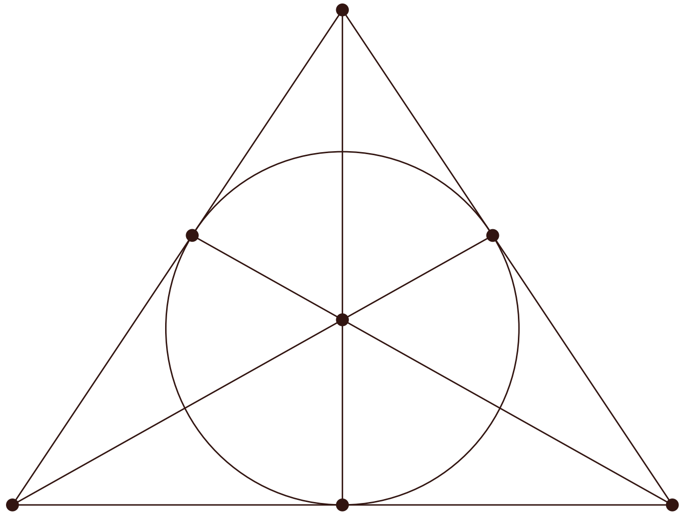
Representations

Definition.

A *representation* of M over field $\mathbb{F}$ is a dependency-preserving map

$$A : E(M) \to \mathbb{F}^r.$$

Example: the Fano matroid



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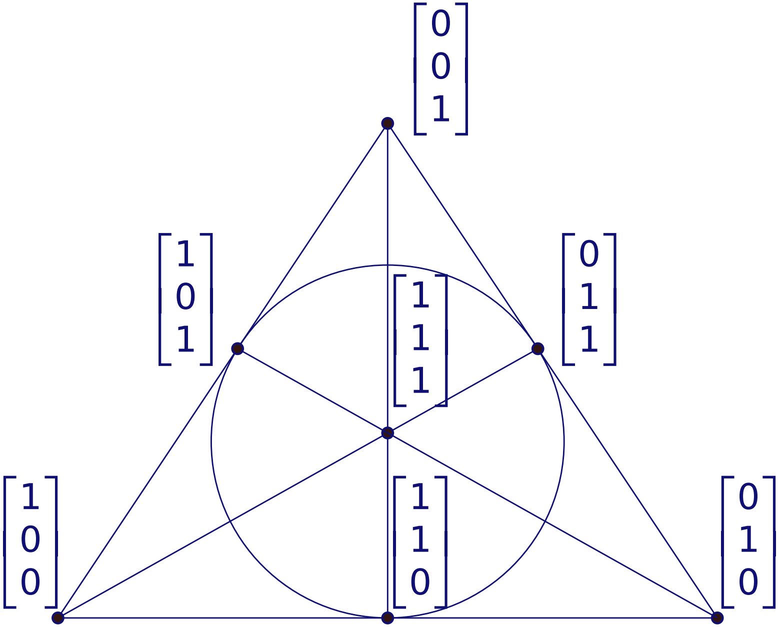

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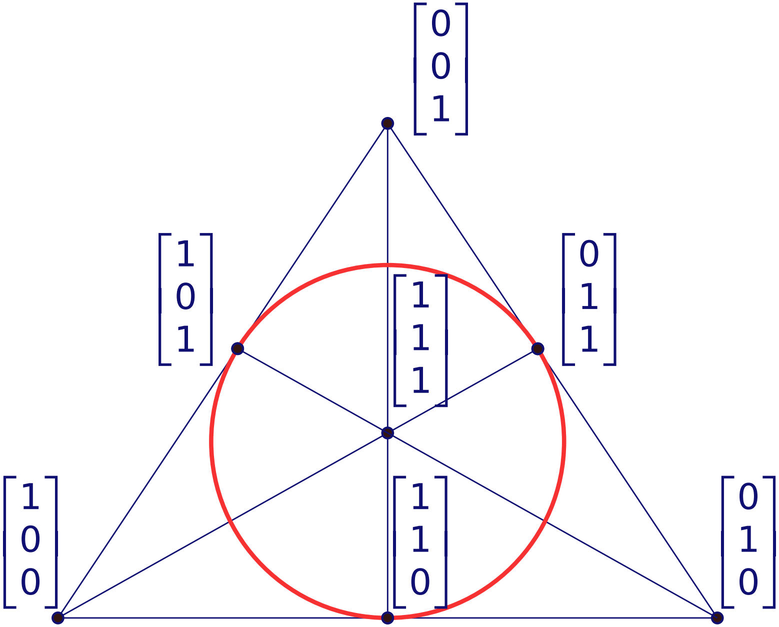


$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = - \begin{bmatrix}0\\0\\2\end{bmatrix}$$

Representations

Definition.
A *representation* of M over field $\mathbb{F}$ is a dependency-preserving map

$$A : E(M) \to \mathbb{F}^r.$$

- View A as *matrix* with columns labeled by E
- Write $M = M[A]$

Regular matroids

Theorem (Tutte 1958).
Equivalent for a matroid M:

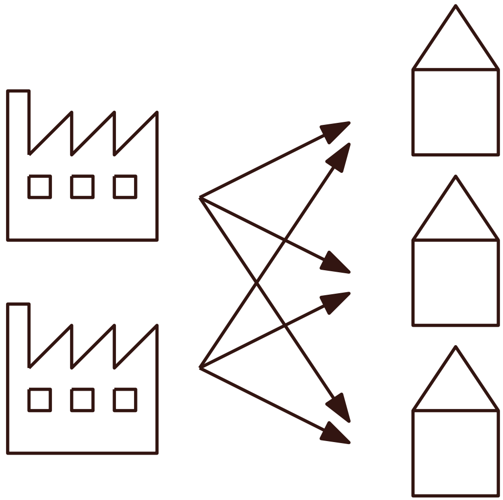
- M representable over *all* fields
- M representable over GF(2) and GF(3)
- M has totally unimodular representation over $\mathbb{R}$

A matrix is *totally unimodular* if every sub-determinant is in

$$\{\pm 1\} \cup \{0\}.$$

Such matroids are called *regular*.

An optimization problem

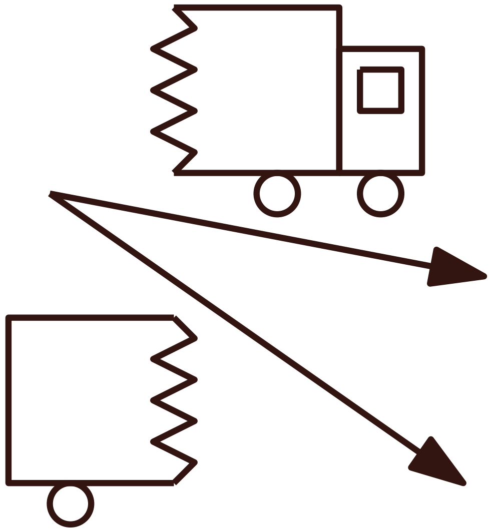


An optimization problem

$$\text{minimize } 2x_{11} + 3x_{12} + 4x_{13} + 2x_{21} + 2x_{22} + 8x_{23}$$

$$\text{such that } \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \\ 3 \end{bmatrix}$$

An optimization problem



An optimization problem

Theorem.
If constraint matrix *totally unimodular*, then integer optimal solution.

. . . and beyond

Theorem (Whittle 1997).
Equivalent for a matroid M:

- M representable over all fields with characteristic $\neq 2$
- M representable over GF(3) and GF(5)
- M has *totally dyadic* representation over $\mathbb{R}$

A matrix is *totally dyadic* if every subdeterminant is in

$$\{\pm 2^k : k \in \mathbb{Z}\} \cup \{0\}.$$

. . . and beyond

Theorem (Whittle 1997).
Equivalent for a matroid M:

- M representable over all fields except, perhaps, GF(2)
- M representable over GF(3), GF(4), GF(5)
- M representable over GF(3), GF(8)
- M has *near-regular* representation over $\mathbb{Q}(\alpha)$

A matrix is *near-regular* if every subdeterminant is in

$$\{\pm\alpha^k(1-\alpha)^l : k,l \in \mathbb{Z}\} \cup \{0\}.$$

. . . and beyond

Theorem (Vertigan, unpublished; Pendavingh, vZ 2010).
Equivalent for a matroid M:

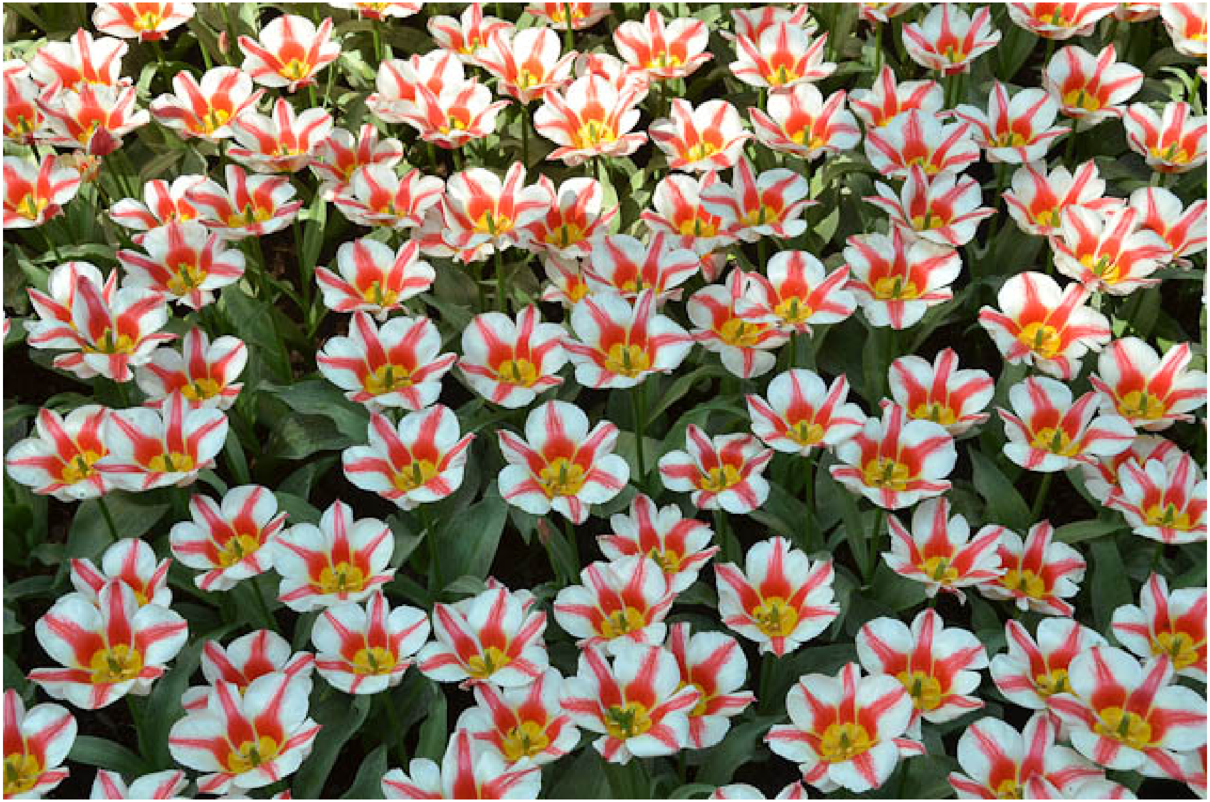
- M representable over GF(4), GF(5)
- M has *totally golden ratio* representation over $\mathbb{R}$

A matrix is *totally golden ratio* if every sub-determinant is in

$$\{\pm\tau^k : k \in \mathbb{Z}\} \cup \{0\}$$

where τ is the *golden ratio*, i.e. $\tau^2 - \tau - 1 = 0$.

Part II
Basis counting



Kirchhoff's Matrix-Tree Theorem

Theorem (Kirchhoff)

Let A be T.U. matrix. Then

$$\det(AA^T) = \#\{B \text{ basis of } M[A]\}$$

Theorem (Cauchy - Binet)

Let A be $r \times s$ matrix; B $s \times r$ matrix. Then

$$\det(AB) = \sum_{|X|=r} \det(A_X)\det(B_X)$$

...and beyond

- *Complex unimodular:* matrix over $\mathbb{C}$, nonzero determinants have norm 1.

$$\det(AA^{\dagger}) = \#\{B \text{ basis of } M[A]\}$$

- *Quaternionic unimodular?*

Problem: determinants only make sense in commutative rings

What Would Tutte Do?

Chain groups

Definition: R ring, E finite set. *Chain group* is

$$C \subseteq R^E$$

such that, for $c, d \in C$ and $r \in R$

- $\underline{0} \in C$
- $c + d \in C$
- $rc \in C$

Definition: *Support* of a chain c:

$$\|c\| := \{e \in E : c_e \neq 0\}$$

Definition: *Elementary chain:* $c \neq \underline{0}$, inclusionwise minimal support.

Chain groups

Definition: *Skew partial field* $\mathbb{P} = (R, G)$

- R ring
- $G \subseteq R^*$ group
- $-1 \in G$

Definition: *G-primitive* chain: $c \in (G \cup \{0\})^E$.

Definition: Chain group is $\mathbb{P}$-chain group if, for all $c \in C$ elementary,

$$c = rd$$

where $r \in R$ and $d \in C$ is G-primitive.

Example:

- *Regular partial field:* $\mathbb{U}_0 = (R, G)$ with
  - $R = \mathbb{Z}$
  - $G = \{-1, 1\}$
- $\mathbb{Z}$-span of rows of T.U. matrix is $\mathbb{U}_0$-chain group

Chain groups

Theorem (Pendavingh, vZ 2009):
For a $\mathbb{P}$-chain group C, define

$$\mathcal{C}^* := \{\|c\| : c \in C, elementary\}$$

Then $\mathcal{C}^*$ is set of cocircuits of a matroid, $M(C)$.

(Co)circuit axioms

$\mathcal{C}^*$ is set of cocircuits of a matroid if and only if

- $\emptyset \notin \mathcal{C}^*$
- $C, D \in \mathcal{C}^*$ and $C \subseteq D$ then $C = D$
- $C, D \in \mathcal{C}^*$, $C \neq D$, $e \in C \cap D$, then $(C \cup D) - e$ contains a cocircuit

Why all this trouble?

- Because we can
- Can represent some matroids that have no representation over any (skew) field
- Captures “multilinear representations” from coding theory
- *Quaternionic Unimodular Matroids*:
  - $R = \mathbb{H}$, the quaternions
  - $G = \{x \in \mathbb{H} : \|x\| = 1\}$

Cauchy-binet extended

Theorem (Pendavingh, vZ 2011+)

Let A be $r \times s$ matrix over $\mathbb{H}$. Then

$$\delta(AA^\dagger) = \sum_{|X|=r} \delta(A_X)\delta(A_X^\dagger)$$

where

$$\delta(D) := \sqrt{|\det(z_2(\varphi(D)))|}$$

Basis counting, extended

$$\delta(AA^\dagger) = \#\{B \text{ basis of } M[A]\}$$

$$P_A := A^\dagger(AA^\dagger)^{-1}A$$

$$\delta(P_A[F,F]) = \frac{\#\{B \text{ basis, } F \subseteq B\}}{\#\{B \text{ basis}\}}$$

Some open problems

Let $\mathbb{P}$ be skew partial field.

- Are $\mathbb{P}$-representable matroids algebraic?
- Does Ingleton's Inequality hold?
- Are there Q.U. matroids not representable over a commutative field?
- Can we get all Q.U. matroids with just a finite subgroup of $\{x \in \mathbb{H} : \|x\| = 1\}$?
- Do Q.U. matroids have the half-plane property?

Part III
Structure

Operations

Elementary operations that preserve T.U.:

- Scale rows and columns by -1
- Permute rows and columns
- Row-reduce a column to an identity vector

$$\left[\begin{array}{c|c} \alpha & c \\ \hline b & D \end{array}\right] \rightarrow \left[\begin{array}{c|c} 1 & \alpha^{-1}c \\ \hline 0 & D - b\alpha^{-1}c \end{array}\right]$$

Operations

Dualizing:

$$[I\ A] \to [-A^T\ I']$$

Operations

Operations that preserve T.U.: 1-sums

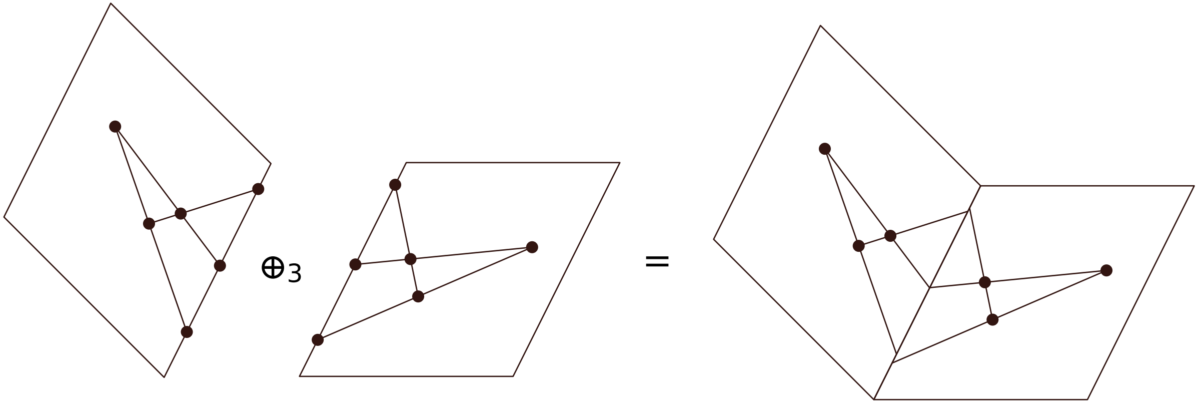
$$A_1 \oplus_1 A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

Operations

Operations that preserve T.U.: 2-sums

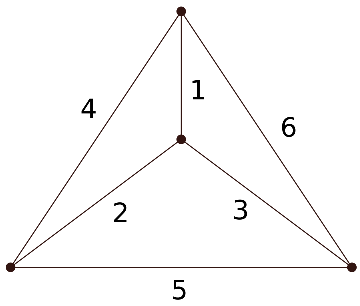
$$\left[\begin{array}{c|c} A_1 & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline a_1 & 1 \end{array}\right] \oplus_2 \left[\begin{array}{c|c} 1 & a_2 \\ \hline \begin{matrix}0\\ \vdots \\ 0\end{matrix} & A_2 \end{array}\right] = \left[\begin{array}{c|c} A_1 & 0 \\ \hline a_1 & a_2 \\ \hline 0 & A_2 \end{array}\right]$$

3-sums



Have cement, need bricks

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$



Theorem.
A graphic matroid is regular.

The case R_{10}

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Theorem. If M regular, contains R_{10}, not equal to R_{10}, it can be written as a 1- or 2-sum.

Seymour's Decomposition Theorem

Theorem (Seymour 1980).
Every regular matroid can be obtained from graphic ones and R_{10} by *dualizing*, *k-sums for* $k = 1, 2, 3$.

Theorem (Tutte 1960 + Seymour 1981).
A matroid can be tested for being graphic in polynomial time.

Theorem (Truemper 1982).
A matroid can be tested for being regular in polynomial time.

. . . and beyond?

Problem.
Can a matroid be tested for being *near-regular* in polynomial time?

Problem.
Is there a satisfying decomposition theorem for near-regular matroids?

Recognizing signed-graphic matroids

Definition.
A matroid is *signed-graphic* $\Leftrightarrow$ representation over GF(3) with at most 2 nonzero entries per column.

Theorem (Geelen; Mayhew – unpublished).
There is no polynomial-time algorithm to test if a matroid, given by rank oracle, is signed-graphic.

But...
What if M is given as GF(3)-matrix?

What about decomposition?

Natural condition for decomposition:

- No basic class contains all graphic *and* all co-graphic matroids.

Corollary (Mayhew, Whittle, vZ 2011).
Any natural decomposition of the near-regular matroids must employ 4-sums.

What about decomposition?

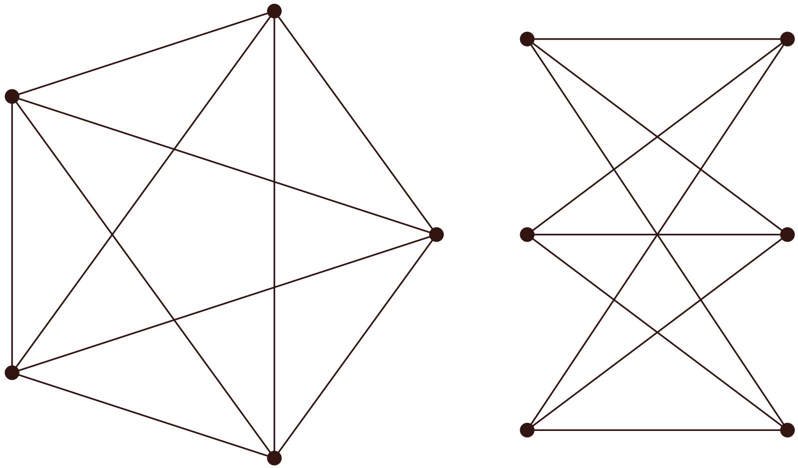
Theorem (Mayhew, Whittle, vZ 2011).
M_1, M_2 graphic matroids. Can build internally 4-connected near-regular matroid having both M_1 and dual of M_2 in it.

$$A_{12} = \begin{array}{c} \\ a \\ b \\ c \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} d & e & f & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & \alpha \\ 1 & 1 & 0 & 0 & \alpha & -\alpha \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

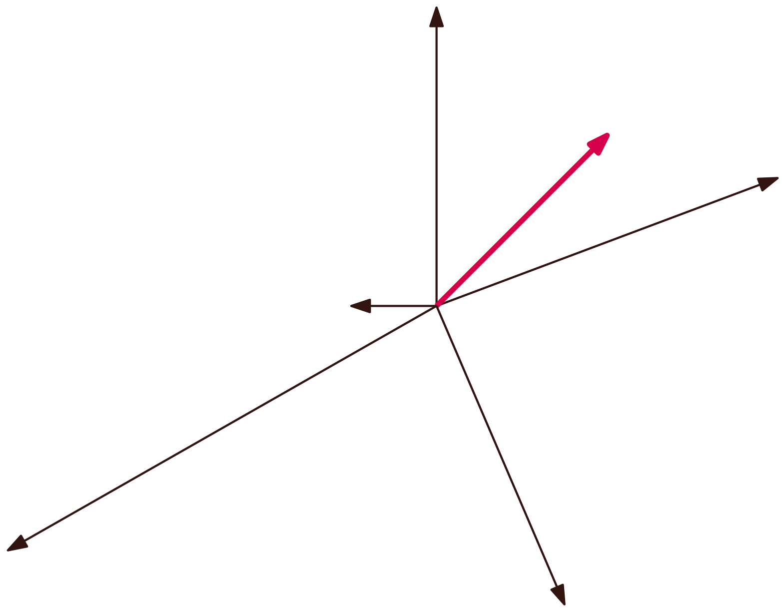
Part IV
Excluded minors

Kuratowski's Theorem

Theorem.
Exactly two excluded minors for planar graphs:



Contraction



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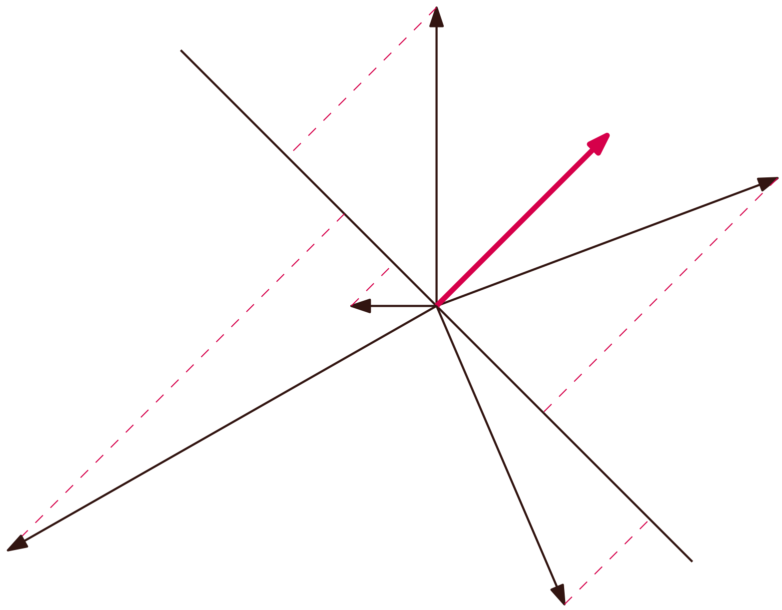

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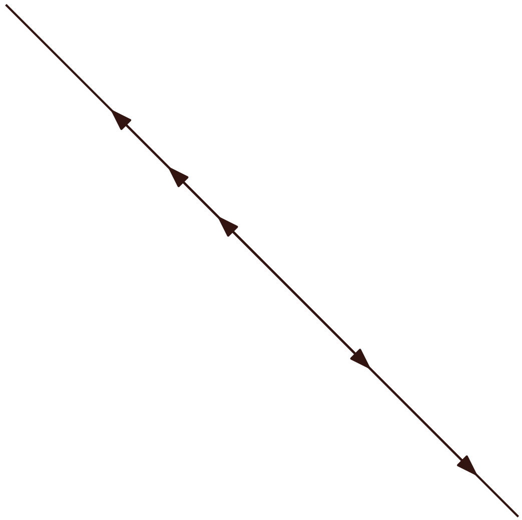


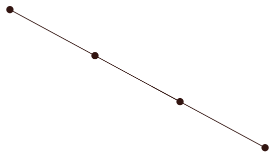
Rota’s Conjecture

Theorem (Tutte 1958):
Exactly 1 excluded minor for

$$\left\{ M : E(M) \rightarrow \right.$$


$$\left. \right\}$$

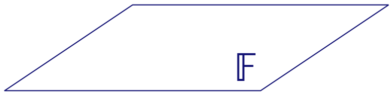
namely

$$\left\{ \right.$$

$$\left. \right\}$$

Rota's Conjecture

Conjecture (Rota 1971): $\mathbb{F}$ finite, then $\exists k = k(\mathbb{F})$: exactly k excluded minors for

$$\left\{ M : E(M) \rightarrow \right.$$



$$\left. \right\}$$

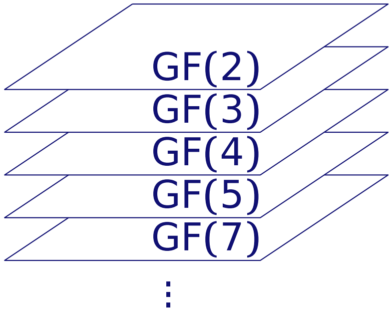
$\mathbb{F}$	GF(2)	GF(3)	GF(4)	GF(5)
k	1	4	7	≥ 564[a]

[a] Mayhew, Royle 2009

Regular matroids

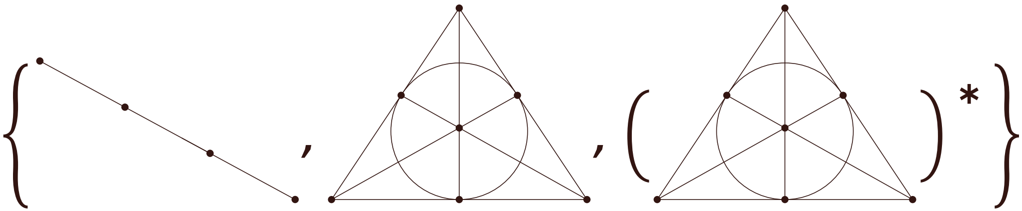
Theorem (Tutte 1958):
Exactly 3 excluded minors for

$$\left\{ M : E(M) \rightarrow \right.$$



$$\left. \right\}$$

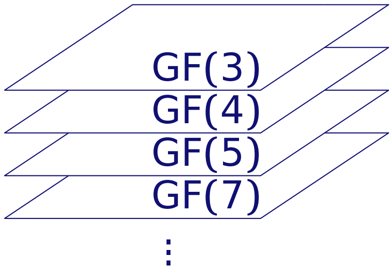
namely



Near-regular matroids

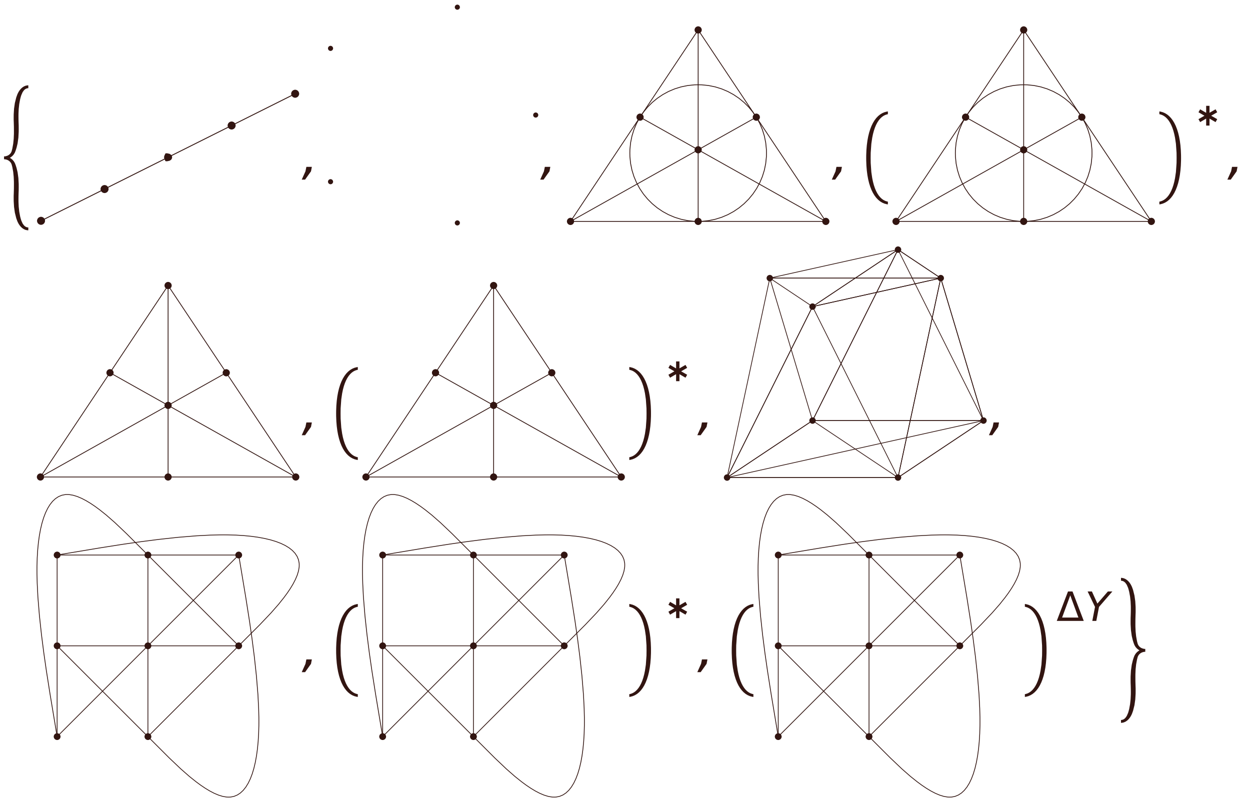
Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

$$\left\{ M : E(M) \to \right.$$



$$\left. \right\}$$

namely



Others?

Sixth-roots-of-unity known.

Major open case: Dyadic matroids.

Slides, papers at
http://www.math.princeton.edu/~svanzwam/

The End