Beyond Total Unimodularity

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The plan:

- Matroid Representations, Whittle’s Classes
- Basis Counting
- Decomposition
- Excluded Minors
Part I
Matroid Representations, Whittle’s Classes
Representations

Definition.
A representation of $M$ over field $\mathbb{F}$ is a dependency-preserving map

$$A : E(M) \rightarrow \mathbb{F}^r.$$
Example: the Fano matroid
Example: the Fano matroid
Example: the Fano matroid

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} - \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} - \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} = -\begin{bmatrix}
0 \\
0 \\
2
\end{bmatrix}
\]
**Representations**

**Definition.**

A representation of $M$ over field $\mathbb{F}$ is a dependency-preserving map

$$A : E(M) \to \mathbb{F}^r.$$

- View $A$ as matrix with columns labeled by $E$
- Write $M = M[A]$
Regular matroids

Theorem (Tutte 1958).
Equivalent for a matroid $M$:

- $M$ representable over all fields
- $M$ representable over $\text{GF}(2)$ and $\text{GF}(3)$
- $M$ has totally unimodular representation over $\mathbb{R}$

A matrix is *totally unimodular* if every sub-determinant is in

$$\{\pm 1\} \cup \{0\}.$$ 

Such matroids are called *regular*. 
An optimization problem
An optimization problem

minimize $2x_{11} + 3x_{12} + 4x_{13} + 2x_{21} + 2x_{22} + 8x_{23}$

such that

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_{11} \\
x_{12} \\
x_{13} \\
x_{21} \\
x_{22} \\
x_{23}
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
6 \\
2 \\
5 \\
3
\end{bmatrix}
$$
An optimization problem
An optimization problem

Theorem.
If constraint matrix *totally unimodular*, then integer optimal solution.
... and beyond

Theorem (Whittle 1997).
Equivalent for a matroid $M$:

- $M$ representable over all fields with characteristic $\neq 2$
- $M$ representable over $\text{GF}(3)$ and $\text{GF}(5)$
- $M$ has totally dyadic representation over $\mathbb{R}$

A matrix is totally dyadic if every subdeterminant is in

$$\{\pm 2^k : k \in \mathbb{Z}\} \cup \{0\}.$$
... and beyond

Theorem (Whittle 1997).
Equivalent for a matroid $M$:

- $M$ representable over all fields except, perhaps, GF(2)
- $M$ representable over GF(3), GF(4), GF(5)
- $M$ representable over GF(3), GF(8)
- $M$ has near-regular representation over $\mathbb{Q}(\alpha)$

A matrix is near-regular if every subdeterminant is in

$$\{\pm \alpha^k (1 - \alpha)^l : k, l \in \mathbb{Z}\} \cup \{0\}.$$
... and beyond

Theorem (Vertigan, unpublished; Pendavingh, vZ 2010).
Equivalent for a matroid $M$:

- $M$ representable over GF(4), GF(5)
- $M$ has *totally golden ratio* representation over $\mathbb{R}$

A matrix is *totally golden ratio* if every sub-determinant is in

$$\{\pm \tau^k : k \in \mathbb{Z}\} \cup \{0\}$$

where $\tau$ is the *golden ratio*, i.e. $\tau^2 - \tau - 1 = 0$. 
Part II

Basis counting
Kirchhoff’s Matrix-Tree Theorem

Theorem (Kirchhoff)
Let $A$ be T.U. matrix. Then
\[
\det(AA^T) = \# \{B \text{ basis of } M[A]\}
\]

Theorem (Cauchy - Binet)
Let $A$ be $r \times s$ matrix; $B$ $s \times r$ matrix. Then
\[
\det(AB) = \sum_{|X|=r} \det(A_X) \det(B_X)
\]
...and beyond

- **Complex unimodular:** matrix over $\mathbb{C}$, nonzero determinants have norm 1.

  $$\det(AA^\dagger) = \# \{ \text{B basis of } M[A] \}$$

- **Quaternionic unimodular?**

**Problem:** determinants only make sense in commutative rings
What Would Tutte Do?
Chain groups

Definition: $R$ ring, $E$ finite set. Chain group is

$$C \subseteq R^E$$

such that, for $c, d \in C$ and $r \in R$

- $0 \in C$
- $c + d \in C$
- $rc \in C$

Definition: Support of a chain $c$:

$$\|c\| := \{e \in E : c_e \neq 0\}$$

Definition: Elementary chain: $c \neq 0$, inclusionwise minimal support.
Chain groups

**Definition:** Skew partial field $\mathbb{P} = (R, G)$

- $R$ ring
- $G \subseteq R^*$ group
- $-1 \in G$

**Definition:** $G$-primitive chain: $c \in (G \cup \{0\})^E$.

**Definition:** Chain group is $\mathbb{P}$-chain group if, for all $c \in C$ elementary,

$$c = rd$$

where $r \in R$ and $d \in C$ is $G$-primitive.
Example:

- Regular partial field: $\mathbb{U}_0 = (R, G)$ with
  - $R = \mathbb{Z}$
  - $G = \{-1, 1\}$
- $\mathbb{Z}$-span of rows of T.U. matrix is $\mathbb{U}_0$-chain group
Chain groups

Theorem (Pendavingh, vZ 2009): For a $\mathcal{P}$-chain group $\mathcal{C}$, define

$$\mathcal{C}^* := \{\|c\| : c \in \mathcal{C}, \text{elementary}\}$$

Then $\mathcal{C}^*$ is set of cocircuits of a matroid, $\mathcal{M}(\mathcal{C})$.

(Co)circuit axioms

$\mathcal{C}^*$ is set of cocircuits of a matroid if and only if

- $\emptyset \notin \mathcal{C}^*$
- $C, D \in \mathcal{C}^*$ and $C \subseteq D$ then $C = D$
- $C, D \in \mathcal{C}^*, C \neq D, e \in C \cap D$, then $(C \cup D) - e$ contains a cocircuit
Why all this trouble?

- Because we can
- Can represent some matroids that have no representation over any (skew) field
- Captures “multilinear representations” from coding theory
- *Quaternionic Unimodular Matroids:*
  - $R = \mathbb{H}$, the quaternions
  - $G = \{x \in \mathbb{H} : \|x\| = 1\}$
Cauchy-binet extended

**Theorem (Pendavingh, vZ 2011+)**

Let $A$ be $r \times s$ matrix over $\mathbb{H}$. Then

$$\delta(AA^\dagger) = \sum_{|X|=r} \delta(A_X)\delta(A_X^\dagger)$$

where

$$\delta(D) := \sqrt{|\det(z_2(\varphi(D)))|}$$
Basis counting, extended

\[ \delta(AA^+) = \# \{ B \text{ basis of } M[A] \} \]

\[ P_A := A^+ (AA^+)^{-1} A \]

\[ \delta(P_A[F,F]) = \frac{\# \{ B \text{ basis, } F \subseteq B \}}{\# \{ B \text{ basis} \}} \]
Some open problems

Let $\mathcal{P}$ be skew partial field.

- Are $\mathcal{P}$-representable matroids algebraic?
- Does Ingleton’s Inequality hold?
- Are there Q.U. matroids not representable over a commutative field?
- Can we get all Q.U. matroids with just a finite subgroup of $\{x \in \mathbb{H} : \|x\| = 1\}$?
- Do Q.U. matroids have the half-plane property?
Part III
Structure
Operations

Elementary operations that preserve T.U.:

- Scale rows and columns by \(-1\)
- Permute rows and columns
- Row-reduce a column to an identity vector

\[
\begin{pmatrix}
\alpha & c \\
b & D
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & \alpha^{-1}c \\
0 & D - b\alpha^{-1}c
\end{pmatrix}
\]
Operations
Dualizing:

\[[I \ A] \rightarrow [-A^T \ I']\]
Operations
Operations that preserve T.U.: 1-sums

\[ A_1 \oplus_1 A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \]
Operations

Operations that preserve T.U.: 2-sums

\[
\begin{bmatrix}
A_1 & 0 \\
\vdots & \vdots \\
a_1 & 0
\end{bmatrix}
\oplus_2
\begin{bmatrix}
1 & a_2 \\
0 & A_2
\end{bmatrix}
= 
\begin{bmatrix}
A_1 & 0 \\
a_1 & a_2 \\
0 & A_2
\end{bmatrix}
\]
3-sums
Have cement, need bricks

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
-1 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 \\
\end{bmatrix}
\]

**Theorem.**
A graphic matroid is regular.
The case $R_{10}$

$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1
\end{bmatrix}
$$

**Theorem.** If $M$ regular, contains $R_{10}$, not equal to $R_{10}$, it can be written as a 1- or 2-sum.
Seymour’s Decomposition Theorem

Theorem (Seymour 1980).
Every regular matroid can be obtained from graphic ones and $R_{10}$ by dualizing, $k$-sums for $k = 1, 2, 3$. 

A matroid can be tested for being graphic in polynomial time.

Theorem (Truemper 1982).
A matroid can be tested for being regular in polynomial time.
... and beyond?

**Problem.**
Can a matroid be tested for being *near-regular* in polynomial time?

**Problem.**
Is there a satisfying decomposition theorem for near-regular matroids?
Recognizing signed-graphic matroids

Definition.
A matroid is signed-graphic ⇔ representation over GF(3) with at most 2 nonzero entries per column.

Theorem (Geelen; Mayhew – unpublished).
There is no polynomial-time algorithm to test if a matroid, given by rank oracle, is signed-graphic.

But...
What if $M$ is given as GF(3)-matrix?
What about decomposition?

Natural condition for decomposition:

- No basic class contains all graphic and all co-graphic matroids.

**Corollary (Mayhew, Whittle, vZ 2011).**

Any natural decomposition of the near-regular matroids must employ 4-sums.
What about decomposition?

**Theorem (Mayhew, Whittle, vZ 2011).**

$M_1, M_2$ graphic matroids. Can build internally 4-connected near-regular matroid having both $M_1$ and dual of $M_2$ in it.

\[
A_{12} = \begin{bmatrix}
    a & b & c & 1 & 2 & 3 \\
    d & e & f & 4 & 5 & 6 \\
    1 & 0 & 1 & 1 & 1 & 0 \\
    0 & -1 & 1 & 1 & 0 & \alpha \\
    1 & 1 & 0 & 0 & \alpha & -\alpha \\
    0 & 0 & 0 & 1 & 0 & 1 \\
    0 & 0 & 0 & 0 & 1 & -1 \\
    0 & 0 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]
Part IV

Excluded minors
Kuratowski’s Theorem

Theorem.
Exactly two excluded minors for planar graphs:
Contraction
Contraction
Contraction
Rota’s Conjecture

Theorem (Tutte 1958):
Exactly 1 excluded minor for

\[ M : E(M) \to \text{GF}(2) \]

namely
Rota’s Conjecture

Conjecture (Rota 1971): $\mathbb{F}$ finite, then $\exists k = k(\mathbb{F})$: exactly $k$ excluded minors for

$$\left\{ M : E(M) \to \mathbb{F} \right\}$$

<table>
<thead>
<tr>
<th>$\mathbb{F}$</th>
<th>GF(2)</th>
<th>GF(3)</th>
<th>GF(4)</th>
<th>GF(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>$\geq 564^a$</td>
</tr>
</tbody>
</table>

$^a$Mayhew, Royle 2009
Regular matroids

Theorem (Tutte 1958):
Exactly 3 excluded minors for

\[ \left\{ M : E(M) \to \{\text{GF}(2), \text{GF}(3), \text{GF}(4), \text{GF}(5), \text{GF}(7), \ldots\} \right\} \]

namely

\[ \left\{ \text{triangle}, \text{triangle}, \left( \text{triangle} \right)^* \right\} \]
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[ M : E(M) \rightarrow \{ \text{GF}(3), \text{GF}(4), \text{GF}(5), \text{GF}(7), \ldots \} \]
namely

\[
\left\{ \ldots, \cdot, \cdot, \text{triangle}, (\text{circle})^*, \right. \\
\left. \text{triangle}, (\text{circle})^*, \Delta Y \right\}
\]
Others?

Sixth-roots-of-unity known.

**Major open case:** Dyadic matroids.
Slides, papers at
http://www.math.princeton.edu/~svanzwam/

The End