Connectivity in graphs and matroids

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Women and Mathematics, Princeton, NJ
May 20, 2013
The plan:
- Menger
- Tutte
- Robertson and Seymour
Part I
Menger
Menger’s Theorem
Menger’s Theorem
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Menger’s Theorem

Theorem (Menger, undirected vertex-disjoint version).

\( G = (V, E) \) undirected graph, \( S, T \subseteq V \).

**Maximum** number of vertex-disjoint \( S – T \) paths

\[ = \]

**Minimum** size of an \( S – T \) cut

Notation (for the minimum): \( \kappa_G(S, T) \).
Menger’s Theorem, variant

Theorem.

$G = (V, E)$ undirected graph, $S, T \subseteq V$, $e \in E$. At least one of the following holds:

- $\kappa_{G\setminus e}(S, T) = \kappa_G(S, T)$
- $\kappa_{G/e}(S, T) = \kappa_G(S, T)$
Menger’s Theorem
Menger’s Theorem
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Menger’s Theorem, variant

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Menger’s Theorem, twice
Menger’s Theorem, twice

**Theorem.**

$G = (V, E)$ undirected graph

$Q, R, S, T \subseteq V$

$k = \kappa_G(Q, R), \quad l = \kappa_G(S, T)$

If $|E| > c(k, l)$ then for some $e \in E$

At least one of the following holds:

- $\kappa_{G\setminus e}(Q, R) = k$ **AND** $\kappa_{G\setminus e}(S, T) = l$
- $\kappa_{G/e}(Q, R) = k$ **AND** $\kappa_{G/e}(S, T) = l$
What’s the best constant?

- Lower bound: $kl$
- For $\min(k, l) \leq 3$: upper bound $10kl$ (Feng Zhu)
- Otherwise: open problem!
Gammoids

Definition.
\( D = (V, A) \) directed graph.
\( S, T \subseteq V \).
\( r = |T| \).

\[ \mathcal{B} = \{ X \subseteq S : \kappa_G(X, T) = r \} \]

The pair \((S, \mathcal{B})\) is a gammoid.

Theorem.
Each gammoid is a matroid.
Gammoids

Research problem.
Find $f : \mathbb{N} \to \mathbb{N}$ such that each gammoid $(S, B)$ with $|S| = n$ has a directed graph with at most $f(n)$ vertices.

$$f(n) \leq 2^2 \underbrace{2 \ldots 2}_{2^{n/2} \text{ times}}$$
Part II
Tutte
Connectivity in graphs

Definition.
A graph is \emph{vertically} $k$-connected if it has no vertex cut of size $< k$. 
Application

Theorem (Whitney).
3-connected planar graphs have a unique plane embedding.
Connectivity à la Tutte

\[ k = |V(G_1) \cap V(G_2)| \]
\[ = |V(G_1)| + |V(G_2)| - |V(G)| \]
\[ = (|V(G_1)| - 1) + (|V(G_2)| - 1) - (|V(G)| - 1) + 1 \]
\[ = r(E_1) + r(E_2) - r(E) + 1 \]
Connectivity à la Tutte

Definition.
Partition \((A, B)\) of \(E(M)\) is \(k\)-separation if \(|A|, |B| \geq k\) and

\[ r(A) + r(B) - r(M) < k \]

Matroid \(M\) is \(k\)-connected if no \(k'\)-separation with \(k' < k\).
Connectivity à la Tutte

Slightly weird:
Menger’s Theorem for Matroids

Definition.

\[ \lambda(X) := r(X) + r(E - X) - r(M) \]
\[ \kappa(S, T) := \min \{ \lambda(X) : S \subseteq X \subseteq E - T \} \]

Tutte’s Linking Theorem.

\( M \) matroid, \( S, T \subseteq E(M) \), \( e \in E(M) \).

At least one of the following holds:

- \( \kappa_{M\setminus e}(S, T) = \kappa_M(S, T) \)
- \( \kappa_{M/e}(S, T) = \kappa_M(S, T) \)
Menger’s Thm for Matroids, variant

**Definition.**

\[
\lambda(X) := r(X) + r(E - X) - r(M) \\
\kappa(S, T) := \min\{\lambda(X) : S \subseteq X \subseteq E - T\}
\]

**Tutte’s Linking Theorem.**

$M$ matroid, $S, T \subseteq E(M), e \in E(M)$.

\[
\max\{\lambda_N(S) : N \preceq M, E(N) = S \cup T\} \\
= \\
\min\{\lambda(X) : S \subseteq X \subseteq E - T\}
\]
Menger’s Thm for Matroids, twice?

Theorem (Huynh, vZ 2013+).

Let $M$ be a representable matroid.

Let $Q, R, S, T \subseteq E$

Let $k = \kappa_M(Q, R)$, $l = \kappa_M(S, T)$

If $|E| > c(k, l)$ then for some $e \in E$

At least one of the following holds:

- $\kappa_{M\backslash e}(Q, R) = k$ \textbf{AND} $\kappa_{M\backslash e}(S, T) = l$
- $\kappa_{M/e}(Q, R) = k$ \textbf{AND} $\kappa_{M/e}(S, T) = l$

Conjecture (Geelen).

Holds for all matroids.
Connectivity and representability

**Theorem (Kahn).**
Let $M$ be a 3-connected quaternary matroid. Any matrices $A, B$ representing $M$ related by:

- Row operations
- Column scaling
- Field automorphism

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & \alpha \\
\end{bmatrix}
\quad \quad \quad
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & \overline{\alpha} \\
\end{bmatrix}
\]
Connectivity and representability

Theorem (Kahn).
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\]
Preserving connectivity

Tutte’s Wheels and Whirls Theorem. Each 3-connected matroid $M$ has an $e \in E(M)$ such that $M \setminus e$ or $M/e$ is 3-connected UNLESS $M$ is a wheel or whirl.
Preserving connectivity

Seymour’s Splitter Theorem.
Each 3-connected $M$ with 3-connected minor $N$ has $e \in E(M)$ such that $M \setminus e$ or $M/e$ is 3-connected and has minor isomorphic to $N$ UNLESS $M$ is a wheel or whirl.
The Splitter Theorem, twice?

**Definition.**
An *intertwine* of matroids $N_1$, $N_2$ is a minor-minimal matroid containing both.

**Question.**
Can we bound size of intertwine?
Part III
Robertson and Seymour
Graph Minors

Graph Minors Theorem, antichain version.
In any infinite sequence of graphs $G_1, G_2, \ldots$ there exist $i, j$ with $G_i \preceq G_j$.

Corollary.
Finitely many graphic intertwines of $G, H$. 
Tangles, the idea
Tangle
Definition.

\[ \lambda(X) = r(X) + r(E - X) - r(M). \]

Definition.
\( \mathcal{T} \) is a tangle of \( M \) of order \( \theta \) if

- \( \lambda(X) < \theta \) \( \Rightarrow \) \( X \in \mathcal{T} \) or \( E - X \in \mathcal{T} \)
- \( X \in \mathcal{T} \) \( \Rightarrow \) \( \lambda(X) < \theta \)
- \( X, Y, Z \in \mathcal{T} \) \( \Rightarrow \) \( X \cup Y \cup Z \neq E(M) \)
- \( E - \{e\} \notin \mathcal{T} \) for all \( e \in E(M) \)
Tangle matroid

Theorem (Geelen, Gerards, Robertson, Whittle).

\[ \rho(X) := \begin{cases} \min \{\lambda(Y) : X \subseteq Y \in \mathcal{T}\} & \text{if } X \subseteq Y \in \mathcal{T} \\ \theta & \text{otherwise} \end{cases} \]

Then \( \rho \) is the rank function of a matroid, \( M(\mathcal{T}) \).

- 2-separating \( X \in \mathcal{T} \) → parallel class
- 3-separating \( X \in \mathcal{T} \) → line
- maximal members → flats
Tangle matroid
Tangle matroid
Tangle matroid

Research question.
If $M$ is representable over $\text{GF}(q)$, is $M(T)$ representable over $\text{GF}(q^k)$?
Algorithmic consequences

- No large tangle $\implies$ thin class of matroids, dynamic programming
- Large tangle $\implies$ large grid minor $\implies$ redundant element
Matroid minors

Graph minors accomplished:

- Structure theorem when excluding a minor.
- No infinite antichains.
- Minor-testing algorithm.

Exciting times: Geelen, Gerards, Whittle are extending this to

matroids representable over \( \text{GF}(q) \)
Slides at
http://www.math.princeton.edu/~svanzwam/
The End