Connectivity in graphs and matroids

Stefan van Zwam

Department of Mathematics
Princeton University

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The plan:

- Menger
- Tutte
- Robertson and Seymour
- Whitney
Part I
Karl Menger (1902 – 1985)
Menger’s Theorem
Menger’s Theorem
Menger’s Theorem
Menger’s Theorem
Menger’s Theorem

Theorem (Menger, undirected vertex-disjoint version).

$G = (V, E)$ undirected graph, $S, T \subseteq V$.

**Maximum** number of vertex-disjoint $S – T$ paths

= **Minimum** size of an $S – T$ cut

Notation (for the minimum): $\kappa_G(S, T)$. 
Minors: deletion \((G \setminus e)\)
Minors: contraction \((G/e)\)
Menger’s Theorem, variant

**Theorem.**

$G = (V, E)$ undirected graph, $S, T \subseteq V$, $e \in E$. At least one of the following holds:

- $\kappa_{G\setminus e}(S, T) = \kappa_G(S, T)$
- $\kappa_{G/e}(S, T) = \kappa_G(S, T)$
Menger’s Theorem
Menger’s Theorem
Menger’s Theorem
Menger’s Theorem, variant

Theorem.

Let $G = (V, E)$ be an undirected graph, $S, T \subseteq V$, and $e \in E$. At least one of the following holds:

- $\kappa_{G \setminus e}(S, T) = \kappa_G(S, T)$
- $\kappa_{G/e}(S, T) = \kappa_G(S, T)$
Menger’s Theorem, twice
Menger’s Theorem, twice

Theorem (Huynh, vZ, sort of).

$G = (V, E)$ undirected graph

$Q, R, S, T \subseteq V$

$k = \kappa_G(Q, R), \quad l = \kappa_G(S, T)$

If $|E| > c(k, l)$ then for some $e \in E$

At least one of the following holds:

- $\kappa_{G \setminus e}(Q, R) = k \ \text{AND} \ \kappa_{G \setminus e}(S, T) = l$
- $\kappa_{G / e}(Q, R) = k \ \text{AND} \ \kappa_{G / e}(S, T) = l$
Menger’s Theorem twice, proof sketch
Menger’s Theorem twice, proof sketch
Menger’s Theorem twice, proof sketch

S

T
Menger’s Theorem twice, proof sketch
Menger’s Theorem twice, proof sketch
Menger’s Theorem twice, proof sketch
Menger’s Theorem twice, proof sketch
What’s the best constant?

- Lower bound: $kl$
- For $\min(k, l) \leq 3$: upper bound $10kl$ (Feng Zhu)
- Otherwise: open problem!
Part II
Bill Tutte (1917 – 2002)
Connectivity in graphs

Definition.
A graph is \textit{vertically $k$-connected} if it has no vertex cut of size $< k$. 
Application

Theorem (Whitney).
3-connected planar graphs have a unique plane embedding.
Application

Theorem (Whitney).

3-connected planar graphs have a unique plane embedding.
Preserving connectivity

Tutte’s Wheels (and Whirls) Theorem.
Each 3-connected simple graph $G$ has $e$ such that $G \setminus e$ or $G/e$ is 3-connected and simple
UNLESS $M$ is a wheel.
Preserving connectivity

Seymour’s Splitter Theorem.
Each 3-connected simple graph $G$ with simple and 3-connected minor $H$
has $e$ such that $G\setminus e$ or $G/e$ is 3-connected, simple, and has minor isomorphic to $H$
UNLESS $G$ is a wheel.
The Splitter Theorem, example

\[ G = \quad H = \]
The Splitter Theorem, example

\[ G = \]

\[ H = \]
The Splitter Theorem, example

\[ G = H = \]
The Splitter Theorem, example

\[ G = \]

\[ H = \]
The Splitter Theorem, example

\[ G = \]

\[ H = \]
The Splitter Theorem, twice?

Definition.
An *intertwine* of graphs $H_1, H_2$ is a minor-minimal graph containing both.

Question.
Can we bound size of intertwine?
Part III

Neil Robertson and Paul Seymour
Graph Minors

Graph Minors Theorem, antichain version.
In any infinite sequence of graphs \( G_1, G_2, \ldots \) there exist \( i, j \) with \( G_i \preceq G_j \).

Corollary.
Finitely many graphic intertwines of \( H_1, H_2 \).
Treewidth, the idea
Treewidth, cops and robbers

Game with $k$ cops, 1 robber. In each round:

- Cops can stay or get into helicopter
- Robber moves in path, avoiding all cops on the ground
- Helicopter cops come back down

Win for cops if robber can’t move eventually; win for robber if he remains uncaptured.
Treewidth, cops and robbers

Game with $k$ cops, 1 robber. In each round:

- Cops can stay or get into helicopter
- Robber moves in path, avoiding all cops on the ground
- Helicopter cops come back down

Win for cops if robber can’t move eventually; win for robber if he remains uncaptured.

If cops win, say \textit{treewidth of }$G$ \textit{is at most }$k - 1$. 
Algorithmic consequences

- Small treewidth $\implies$ thin class of graphs, dynamic programming
- Large treewidth $\implies$ large grid minor $\implies$ redundant vertex
Matroid minors

Graph minors accomplished:

- Structure theorem when excluding a minor.
- No infinite antichains.
- Minor-testing algorithm.
Matroid minors

Graph minors accomplished:

- Structure theorem when excluding a minor.
- No infinite antichains.
- Minor-testing algorithm.

Exciting times: Geelen, Gerards, Whittle are extending this to

matroids representable over $\text{GF}(q)$
Part IV
Hassler Whitney (1907 – 1989)
Graphs and matroids
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Graphs and matroids
Graphs and matroids
**Matroid axioms**

**Lemma.** Given

- \( E \): finite set of vectors
- \( \mathcal{I} \): collection of linearly independent subsets

then

- \( \emptyset \in \mathcal{I} \)
- \( J \in \mathcal{I} \) and \( I \subseteq J \), then \( I \in \mathcal{I} \)
- \( I, J \in \mathcal{I} \) and \( |I| < |J| \), then

\[ \exists e \in J - I \text{ such that } I \cup \{e\} \in \mathcal{I} \]
Matroid axioms

**Definition.** Given

- $E$: finite set
- $\mathcal{I}$: collection of subsets

such that

- $\emptyset \in \mathcal{I}$
- $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- $I, J \in \mathcal{I}$ and $|I| < |J|$, then
  $$\exists e \in J - I \text{ such that } I \cup \{e\} \in \mathcal{I}$$

Then $M = (E, \mathcal{I})$ is a **matroid**.
Minors: deletion

\[
\begin{array}{c}
\text{Original graph:} \\
\begin{tikzpicture}
  \node (u) at (0,0) {$u$};
  \node (v) at (2,0) {$v$};
  \draw (u) -- (v) node [midway, anchor=center] {$e$};
\end{tikzpicture}
\end{array}
\rightarrow
\begin{array}{c}
\text{Graph after deletion:} \\
\begin{tikzpicture}
  \node (u) at (0,0) {$u$};
  \node (v) at (2,0) {$v$};
\end{tikzpicture}
\end{array}
\]
Minors: contraction

$u$ $v$

$e$

$uv$
Minors: deletion
Minors: deletion
Minors: contraction
Minors: contraction
Minors: contraction
Menger’s Theorem for Matroids

Definition.

\[ \lambda(X) := r(X) + r(E - X) - r(M) \]
\[ \kappa(S, T) := \min \{ \lambda(X) : S \subseteq X \subseteq E - T \} \]

Tutte’s Linking Theorem.

\( M \) matroid, \( S, T \subseteq E(M), e \in E(M) \).

At least one of the following holds:

- \( \kappa_{M \setminus e}(S, T) = \kappa_M(S, T) \)
- \( \kappa_{M/e}(S, T) = \kappa_M(S, T) \)
Menger’s Thm for Matroids, twice?

**Theorem (Huynh, vZ 2013+).**

*M* representable matroid  

\[ Q, R, S, T \subseteq E \]

\[ k = \kappa_M(Q, R), \quad l = \kappa_M(S, T) \]

If \(|E| > c(k, l)|\) then for some \(e \in E\)

At least one of the following holds:

- \(\kappa_{M\setminus e}(Q, R) = k \ \text{AND} \ \kappa_{M\setminus e}(S, T) = l\)
- \(\kappa_{M/e}(Q, R) = k \ \text{AND} \ \kappa_{M/e}(S, T) = l\)

**Conjecture (Geelen).**

Holds for all matroids.
Preserving connectivity

Tutte’s Wheels and Whirls Theorem. Each 3-connected matroid $M$ has an $e \in E(M)$ such that $M \setminus e$ or $M/e$ is 3-connected UNLESS $M$ is a wheel or whirl.
Preserving connectivity

Seymour’s Splitter Theorem.
Each 3-connected $M$ with 3-connected minor $N$ has $e \in E(M)$ such that $M \setminus e$ or $M/e$ is 3-connected and has minor isomorphic to $N$ UNLESS $M$ is a wheel or whirl.
Removing lots of elements

Theorem (Geelen, vZ 2010+)
Given $k$, if treewidth high enough then have $X$ of size $k$ such that $M\setminus X$ or $M/X$ is 3-connected with $N$-minor.
Slides at
http://www.math.princeton.edu/~svanzwam/

The End