Templates for classes of representable matroids

Stefan van Zwam
Joint work with Kevin Grace

Department of Mathematics
Louisiana State University

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Part I
Structure of minor-closed classes
Graph Minors Structure Theorem

Theorem (Robertson and Seymour 2003).
Let \( \mathcal{G} \) be proper minor-closed class of graphs. Each \( G \in \mathcal{G} \) admits a *tree-decomposition*, whose parts are almost embeddable in a surface.
Graph Minors Structure Theorem

Theorem (Robertson and Seymour 2003).
Let $\mathcal{G}$ be proper minor-closed class of graphs. Each $G \in \mathcal{G}$ admits a \textit{tree-decomposition}, whose parts are almost embeddable in a surface.

Consequences:

- No infinite antichains of graphs;
- $\mathcal{G}$ has finite set of excluded minors;
- Algorithms.
Matroid minors: the blueprint

Theorem (Seymour 1980).
Let $M$ be a regular matroid. Then $M$ can be constructed from graphic matroids, cographic matroids, and $R_{10}$ through 1-, 2-, 3-sums.
Matroid minors: the blueprint

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Highly connected regular matroids are:

- Graphic matroids
- Cographic matroids

What can happen for other classes?
Matroid Minors Structure Theorem

Hypothesis (Geelen, Gerards, Whittle). Let $\mathcal{M}$ be proper minor-closed class of matroids representable over $\text{GF}(q)$. Each $M \in \mathcal{M}$ admits a tree-decomposition, whose parts are

- almost frame matroids; or
- duals of almost frame matroids; or
- almost representable over a subfield of $\text{GF}(q)$.
Constructions

Even-cycle matroids

\[
\begin{array}{cccc}
* & * & \cdots & * \\
\end{array}
\]

\[\leq 2 \text{ ones per column}\]
Constructions

Grafts

\[
\begin{array}{c|c}
\leq 2 \text{ ones per column} & * \\
\vdots & \\
* & \\
\end{array}
\]
Constructions

Grafts

\[
\begin{array}{c}
\leq 2 \text{ ones per column} \\
\vdots \\
\ast
\end{array}
\]

Close under minors: duals of even-cut matroids (Guenin, Pivotto, Wollan).
Constructions

Almost representable over a subfield
A matroid is GF(q)-regular if it is representable over GF(q^t) for all t ≥ 2.

Theorem (Nelson, vZ 2015)
If M is highly connected and has a large PG(t, q) minor, then equivalent:

• M is GF(q)-regular;
• M is representable over GF(q^2) and GF(q^t) for some t ≥ 3;
• M is a restriction of PG(r − 1, q) or PG(r − 1, q).
Perturbations

Definition.
A rank-$(\leq t)$ perturbation of $M = M[A]$ is the matroid $M[A + P]$, where $P$ has matrix rank $\leq t$. 
**Perturbations**

**Definition.**
A rank-\((\leq t)\) perturbation of \(M = M[A]\) is the matroid \(M[A + P]\), where \(P\) has matrix rank \(\leq t\).
Matroid Minors Structure Theorem

Hypothesis (Geelen, Gerards, Whittle 2015). Let $\mathcal{M}$ be proper minor-closed class of matroids representable over $GF(q)$. There exist $k, t, l$ such that each \textit{vertically} $k$-connected $M \in \mathcal{M}$ of size $\geq l$ is

- rank-$(\leq t)$ perturbation of frame matroid; or
- dual of rank-$(\leq t)$ perturbation of frame matroid; or
- rank-$(\leq t)$ perturbation of matroid representable over a subfield.
## Templates

**Frame template, binary version:**

<table>
<thead>
<tr>
<th>$X$</th>
<th>columns from $\Lambda$</th>
<th>0</th>
<th>$A_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq 2$ ones per column</td>
<td>unit columns</td>
<td>rows from $\Delta$</td>
<td></td>
</tr>
</tbody>
</table>
### Templates

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#### Definition.

\( \mathcal{M}(\Phi) \) is set of matroids built from template \( \Phi \).
Templates

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<table>
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<tr>
<th></th>
<th>Z</th>
<th>Y₁</th>
<th>Y₀</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>columns from Λ</td>
<td>0</td>
<td></td>
<td>A₁</td>
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Definition.
\( \mathcal{M}(Φ) \) is set of matroids built from template \( Φ \).

Perturbation Hypothesis (implicit in GGW ’15)
Every perturbation can be built from a template.
Other hypotheses

Let $\mathcal{M}$ be a minor-closed class of $GF(q)$-representable matroids.

**Template List Hypothesis (GGW ’15)**

If $\mathcal{M}(\Phi) \subseteq \mathcal{M}$, and highly connected matroids conform or coconform to $\Phi$, then $\Phi$ is equivalent to one of finitely many templates.
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**Template Covering Hypothesis (GGW ’15)**

All vertically $k$-connected matroids conform or coconform to a template from this list.
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**Template Covering Hypothesis (GGW ’15)**
All vertically \( k \)-connected matroids conform or coconform to a template from this list.

**Growth Rate Hypothesis (Grace, vZ; based on GGW ’15 and Geelen and Nelson 2015)**
The extremal function (growth rate function) of \( \mathcal{M} \) is attained infinitely often by a matroid conforming to a template.
Part II
Applications
Creating an order on templates

Theorem (Grace, vZ 2017).
For every binary frame template $\Phi$, one of the following holds:

- $\Phi$ is trivial;
- $\Phi$ reduces to $\Phi_C$, $\Phi_D$, $\Phi_{CD}$, $\Phi_{Y_0}$, or $\Phi_{Y_1}$;
- For some $k, l$ no simple, vertically $k$-connected matroid of size $\geq l$ conforms or coconforms to $\Phi$.

<table>
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Creating an order on templates

\[ \Phi_0 : \]
\[ \leq 2 \text{nonzeroes per col} \]

\[ \Phi_D : \]
\[ \leq 2 \text{nonzeroes per col} \]

\[ \Phi_{C/\Phi_Y} : \]
\[ \leq 2 \text{nonzeroes per col} \]

\[ \Phi_{CD} : \]
\[ \leq 2 \text{nonzeroes per col} \]

\[ \Phi_{Y_1} : \]
\[ \leq 2 \text{nonzeroes per col} \]

\[ \begin{array}{cccc}
1 & \cdots & 1 & 0 \cdots 0 \\
0 & \cdots & 0 & 1 \cdots 1 \end{array} \]

\[ \begin{array}{cccc}
1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & 0 \\
1 & \cdots & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
\end{array} \]
Applications: 1-flowing matroids

Conjecture (Seymour 1981).
The excluded minors for 1-flowing matroids are $U_{2,4}$, $AG(3, 2)$, $T_{11}$, $T_{11}^*$. 
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Theorem (Grace, vZ 2017).
The template list for 1-flowing matroids is \( \{ \Phi_0 \} \).
Applications: 1-flowing matroids

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Theorem (Grace, vZ 2017).
The template list for 1-flowing matroids is $\{\Phi_0\}$.

Corollary (Grace, vZ 2017).
Subject to Template Covering Hypothesis, a counterexample to Seymour’s 1-Flowing Conjecture has low-order separation or small size.
Applications: fewer excluded minors

**Theorem (Wagner).**
A 3-connected graph on $\geq 11$ edges is planar if and only if it has no $K_{3,3}$-minor.
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Theorem (Grace, vZ 2017+).
Subject to Template Covering Hypothesis, a highly connected binary matroid is in $EX(PG(3, 2) \setminus e, M^*(K_6), L_{11})$ if and only if it is even-cycle.
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Subject to Template Covering Hypothesis, a highly connected binary matroid is in \( EX(PG(3, 2) \setminus e, M^*(K_6), L_{11}) \) if and only if it is even-cycle.

Similar results for:
- Even-cycle with blocking pair
- Even-cut
- Ternary signed-graphic (work in progress)
- Dyadic (work in progress)
Applications: growth rates

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Subject to Growth Rate Hypothesis:

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$$h_{EX(PG(3,2))} \approx r^2 - r + 1.$$ 

Theorem (Grace, work in progress).
Let $\mathcal{G}$ be class of Golden Ratio matroids. Subject to Growth Rate Hypothesis:

$$h_{\mathcal{G}} \approx \binom{r + 3}{2} - 5,$$

verifying a conjecture by Archer (2005) for sufficiently high ranks.
Applications: 2-regular matroids

Theorem (Grace, work in progress).
Let $M$ be highly connected, representable over GF(4) and fields of all characteristics. Then:

- $M$ is representable over all fields with $\geq 4$ elements (2-regular); or
- $M$ is representable over GF(4) and GF($q$) for $q \geq 7$; or
- $M$ is Golden Ratio
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Note: without connectivity assumption, infinitely many classes (Whittle 2005).
Applications: approach

Verify Template List Hypothesis for a class $\mathcal{M}$: explicitly find all templates $\Phi$ such that $\mathcal{M}(\Phi) \subseteq \mathcal{M}$. To rule out a potential template:

- Show $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi')$ with $\Phi'$ in list; or
- Show the matroids conforming to the template are not highly connected; or
- Find certificate placing it outside $\mathcal{M}$.

Typically, try to build an excluded minor for $\mathcal{M}$ using $\Phi$. 
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Note: This procedure yields theorems, independent of hypotheses!
Part III

A Speed Bump
Trouble in template paradise?

Theorem? (Geelen, Gerards, Whittle 2015). Let $\mathcal{M}$ be proper minor-closed class of matroids representable over $\text{GF}(q)$. There exist $k, t, l$ such that each vertically $k$-connected $M \in \mathcal{M}$ of size $\geq l$ is

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- dual of rank-$(\leq t)$ perturbation of frame matroid; or
- rank-$(\leq t)$ perturbation of matroid represented over a subfield.
The counterexample

Let $\mathcal{D}$ be class of *dyadic* matroids (i.e. representable over GF(3) and GF(5)).

‘Theorem’ (Grace, vZ 2017+).

For each $k, t, l$ there exists a vertically $k$-connected dyadic matroid $M$ on $\geq l$ elements, such that **NO** rank-$(\leq t)$ perturbation is a represented frame matroid or the dual of a represented frame matroid.
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Consolation:

- Vertical $k$-connectivity and cographic don’t mix;
- Most results saved by going to cyclic $k$-connectivity when “almost dual of frame”;
- Everything should hold when the word “vertically” is struck out.
Slides, articles at
http://www.math.lsu.edu/~svanzwam/

The End

Stefan van Zwam
When Matroids are Highly Connected