Templates for classes of representable matroids

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Part I
Structure of minor-closed classes

Graph Minors Structure Theorem

Theorem (Robertson and Seymour 2003).
Let $\mathcal{G}$ be proper minor-closed class of graphs. Each $G \in \mathcal{G}$ admits a *tree-decomposition*, whose parts are *almost embeddable in a surface*.

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Consequences:

- No infinite antichains of graphs;
- $\mathcal{G}$ has finite set of excluded minors;
- Algorithms.

Matroid minors: the blueprint

Theorem (Seymour 1980).
Let M be a *regular* matroid. Then M can be constructed from graphic matroids, cographic matroids, and R_{10} through 1-, 2-, 3-sums.

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Highly connected regular matroids are:

- Graphic matroids
- Cographic matroids

What can happen for other classes?

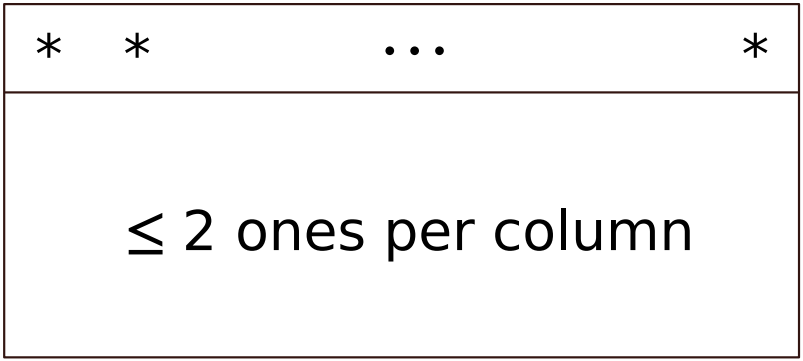
Matroid Minors Structure Theorem

Hypothesis (Geelen, Gerards, Whittle).
Let $\mathcal{M}$ be proper minor-closed class of matroids representable over GF(q). Each $M \in \mathcal{M}$ admits a *tree-decomposition*, whose parts are

- *almost frame matroids*; or
- *duals of almost frame matroids*; or
- *almost representable over a subfield of* GF(q).

Constructions

Even-cycle matroids



Constructions

Grafts

≤ 2 ones per column	$*$ $\vdots$ $*$

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Close under minors: duals of *even-cut matroids* (Guenin, Pivotto, Wollan).

Constructions

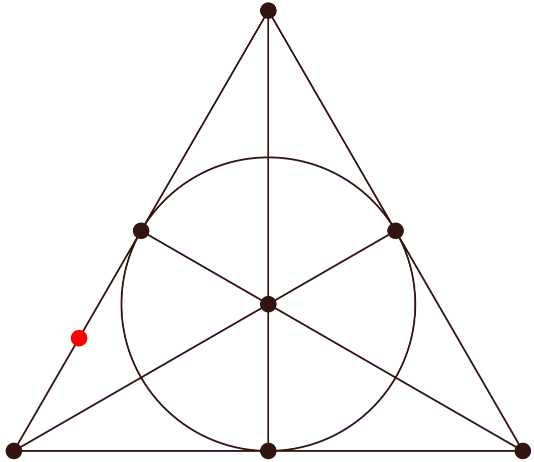
Almost representable over a subfield

A matroid is GF(q)-regular if it is representable over GF(q^t) for all $t \geq 2$.

Theorem (Nelson, vZ 2015)

If M is highly connected and has a large PG(t,q) minor, then equivalent:

- M is GF(q)-regular;
- M is representable over GF(q^2) and GF(q^t) for some $t \geq 3$;
- M is a restriction of $\widehat{\mathrm{PG}}(r-1,q)$ or $\overline{\mathrm{PG}}(r-1,q)$.



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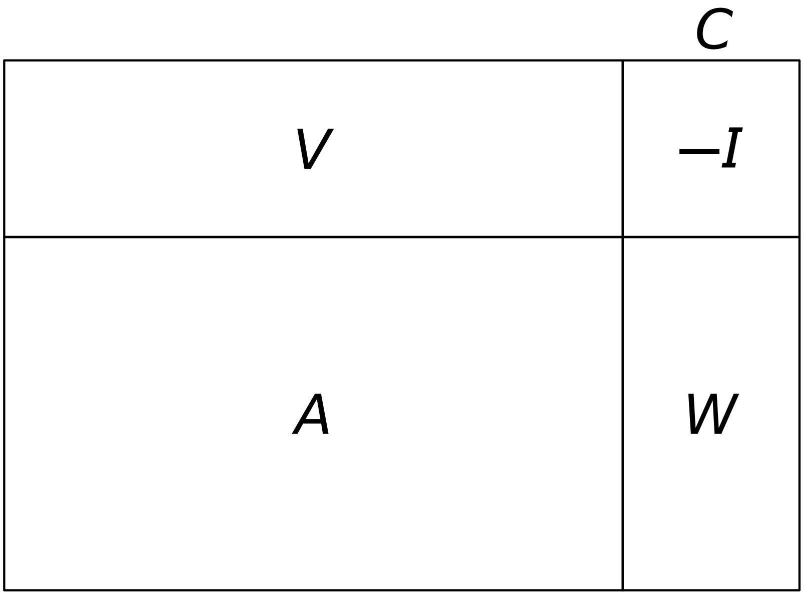

Matroid Minors Structure Theorem

Hypothesis (Geelen, Gerards, Whittle 2015). Let $\mathcal{M}$ be proper minor-closed class of matroids representable over GF(q). There exist k, t, l such that each vertically k-connected $M \in \mathcal{M}$ of size $\geq l$ is

- rank-($\leq t$) perturbation of frame matroid; or
- dual of rank-($\leq t$) perturbation of frame matroid; or
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Templates

Frame template, binary version:

		Z	Y_1 Y_0 C
X	columns from Λ	0	A_1
	≤ 2 ones per column	unit columns	rows from Δ

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Perturbation Hypothesis (implicit in GGW '15)
Every perturbation can be built from a template.

Other hypotheses

Let $\mathcal{M}$ be a minor-closed class of GF(q)-representable matroids.

Template List Hypothesis (GGW '15)

If $\mathcal{M}(\Phi) \subseteq \mathcal{M}$, and highly connected matroids conform or coconform to Φ, then Φ is equivalent to one of finitely many templates.

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Growth Rate Hypothesis (Grace, vZ; based on GGW ’15 and Geelen and Nelson 2015)

The *extremal function* (growth rate function) of $\mathcal{M}$ is attained infinitely often by a matroid conforming to a template.

Part II
Applications

Creating an order on templates

Theorem (Grace, vZ 2017).
For every binary frame template Φ, one of the following holds:

- Φ is trivial;
- Φ reduces to Φ_C, Φ_D, Φ_{CD}, Φ_{Y_0}, or Φ_{Y_1};
- For some k, l no simple, vertically k-connected matroid of size $\geq l$ conforms or coconforms to Φ.

		Z	Y_1 Y_0 C
X	columns from Λ	0	A_1
	≤ 2 ones per column	unit columns	rows from Δ

Creating an order on templates

Φ_0 :

$\leq$ 2nonzeroes per col

Φ_D :

$\underline{x}$
$\leq$ 2nonzeroes per col

Φ_C/Φ_{Y_0}

$\leq$ 2nonzeroes per col	$\underline{y}$

Φ_{CD}

$\underline{x}$	1
$\leq$ 2nonzeroes per col	$\underline{y}$

Φ_{Y_1}

	$1 \cdots 1$ $0 \cdots 0$	$0 \cdots 0$ $1 \cdots 1$	$1 \cdots 1$ $1 \cdots 1$	1 0 1 0 1 1
$\leq$ 2nonzeroes per col	I	I	I	0

Applications: 1-flowing matroids

Conjecture (Seymour 1981).

The excluded minors for 1-flowing matroids are $U_{2,4}$, $\mathrm{AG}(3,2)$, T_{11}, T_{11}^*.

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Corollary (Grace, vZ 2017).
Subject to Template Covering Hypothesis, a counterexample to Seymour's 1-Flowing Conjecture has low-order separation or small size.

Applications: fewer excluded minors

Theorem (Wagner).
A 3-connected graph on ≥ 11 edges is planar if and only if it has no $K_{3,3}$-minor.

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Theorem (Grace, vZ 2017+).
Subject to Template Covering Hypothesis, a highly connected binary matroid is in $\text{EX}(\text{PG}(3,2)\backslash e, M^*(K_6), L_{11})$ if and only if it is even-cycle.

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Similar results for:

- Even-cycle with *blocking pair*
- Even-cut
- Ternary signed-graphic (work in progress)
- Dyadic (work in progress)

Applications: growth rates

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Theorem (Grace, work in progress).
Let $\mathcal{G}$ be class of *Golden Ratio matroids*. Subject to Growth Rate Hypothesis:

$$h_{\mathcal{G}} \approx \binom{r+3}{2} - 5,$$

verifying a conjecture by Archer (2005) for sufficiently high ranks.

Applications: 2-regular matroids

Theorem (Grace, work in progress).
Let M be highly connected, representable over GF(4) and fields of all characteristics. Then:

- M is representable over all fields with ≥ 4 elements (2-regular); or
- M is representable over GF(4) and GF(q) for $q \geq 7$; or
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Note: without connectivity assumption, infinitely many classes (Whittle 2005).

Applications: approach

Verify Template List Hypothesis for a class $\mathcal{M}$: explicitly find all templates Φ such that $\mathcal{M}(\Phi) \subseteq \mathcal{M}$. To rule out a potential template:

- Show $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi')$ with Φ' in list; or
- Show the matroids conforming to the template are not highly connected; or
- Find certificate placing it outside $\mathcal{M}$.
  - Typically, try to build an *excluded minor for* $\mathcal{M}$ using Φ.

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Note: This procedure yields theorems, independent of hypotheses!

Part III
A Speed Bump

Trouble in template paradise?

Theorem? (Geelen, Gerards, Whittle 2015).
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The counterexample

Let $\mathcal{D}$ be class of *dyadic* matroids (i.e. representable over GF(3) and GF(5)).

'Theorem' (Grace, vZ 2017+).

For each k, t, l there exists a vertically k-connected dyadic matroid M on $\geq l$ elements, such that **NO** rank-($\leq t$) perturbation is a represented frame matroid or the dual of a represented frame matroid.

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Consolation:

- Vertical k-connectivity and cographic don't mix;
- Most results saved by going to cyclic k-connectivity when "almost dual of frame";
- Everything should hold when the word "vertically" is struck out.

Slides, articles at
http://www.math.lsu.edu/~svanzwam/

The End