A Stroll through Partial Fields

Stefan van Zwam

Department of Mathematics
Louisiana State University

Binghamton University, April 2, 2018
Part I
Whittle’s Partial Fields
Regular matroids

Theorem (Tutte 1958).
Equivalent for a matroid $M$:

- $M$ representable over all fields
- $M$ representable over GF(2) and GF(3)
- $M$ has totally unimodular representation over $\mathbb{R}$

A matrix is \textit{totally unimodular} if every subdeterminant is in

$$\{\pm 1\} \cup \{0\}.$$ 

Such matroids are called \textit{regular}. 
...and beyond

Theorem (Whittle 1997).
Equivalent for a matroid $M$:

- $M$ representable over all fields with characteristic $\neq 2$
- $M$ representable over $\text{GF}(3)$ and $\text{GF}(5)$
- $M$ has \textit{dyadic} representation over $\mathbb{R}$

A matrix is \textit{dyadic} if every subdeterminant is in

$$\{\pm 2^k : k \in \mathbb{Z}\} \cup \{0\}.$$
and beyond

Theorem (Whittle 1997). Equivalent for a matroid \( M \):

- \( M \) representable over all fields except, perhaps, \( \mathbb{F}(2) \)
- \( M \) representable over \( \mathbb{F}(3) \), \( \mathbb{F}(4) \), \( \mathbb{F}(5) \)
- \( M \) representable over \( \mathbb{F}(3) \), \( \mathbb{F}(8) \)
- \( M \) has near-regular representation over \( \mathbb{Q}(\alpha) \)

A matrix is near-regular if every subdeterminant is in

\[ \{ \pm \alpha^k(1 - \alpha)^l : k, l \in \mathbb{Z} \} \cup \{0\} \]
and beyond

Theorem (Whittle 1997).
Equivalent for a matroid $M$:

- $M$ representable over $\mathrm{GF}(3)$, $\mathrm{GF}(4)$
- $M$ has *sixth-roots-of-unity* representation over $\mathbb{C}$

A matrix is *sixth-roots-of-unity* if every subdeterminant is in

$$\{\zeta^k : k \in \mathbb{Z}\} \cup \{0\}$$

where $\zeta$ is a primitive sixth root of unity, i.e. $\zeta^2 - \text{zeta} + 1 = 0$. 
Whittle’s Classification

Theorem (Whittle 1997).

Let $\mathcal{M}$ be class of matroids representable over $\text{GF}(3)$ and $\mathcal{F}$, where $\mathcal{F}$ does not have characteristic 3. Then $\mathcal{M}$ is the class of matroids representable over $\text{GF}(3)$ and $\text{GF}(q)$ for some

$$q \in \{2, 4, 5, 7, 8\}.$$
Theorem (Vertigan, unpublished; Pendavingh, vZ 2010).
Equivalent for a matroid $M$:
- $M$ representable over $GF(4)$, $GF(5)$
- $M$ has *golden ratio* representation over $\mathbb{R}$

A matrix is *golden ratio* if every subdeterminant is in

$$\{ \pm \tau^k : k \in \mathbb{Z} \} \cup \{0\}$$

where $\tau$ is the *golden ratio*, i.e. $\tau^2 - \tau - 1 = 0$. 
Partial fields

Goal:

- Algebraic structure $\mathbb{P}$
- Matroid represented by matrix over $\mathbb{P}$
- Subdeterminants in $S \cup \{0\}$ where $S \subseteq \mathbb{P}$

First attempt

A partial field is a set with a $\cdot$ operator and partial $+$ operator, subject to some axioms.

Theorem (vZ, Pendavingh 2010).
Need more complicated associative law to make axiomatic approach work.
Partial fields

Definition (Pendavingh, vZ 2010).
Partial field $\mathbb{P}$ is a pair $(R, G)$ of a commutative ring $R$ and subgroup $G$ of $R^*$ such that $-1 \in G$.

Definition.
Matrix $A$ is $\mathbb{P}$-matrix if, for all square submatrices $D$,

$$\det(D) \in G \cup \{0\}.$$ 

Matroid $M$ is $\mathbb{P}$-representable if $M = M(A)$ for some $\mathbb{P}$-matrix (bases $\leftrightarrow$ non-zero maximal subdeterminants).

Theorem.
$M(A)$ is a matroid.
Partial fields
Some examples of partial fields:

**Regular** \((\mathbb{Z}, \{-1, 1\})\)

**Dyadic** \((\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle)\)

**Sixth-roots-of-unity** \((\mathbb{Z}[\zeta], \langle \zeta \rangle)\), where \(\zeta^2 - \zeta + 1 = 0\)

**Near-regular** \((\mathbb{Z}[\alpha, \frac{1}{\alpha}, \frac{1}{1-\alpha}], \langle -1, \alpha, 1-\alpha \rangle)\)

**Golden ratio** \((\mathbb{Z}[\tau], \langle -1, \tau \rangle)\)

**Product partial field** \((F_1 \times F_2, (F_1 \times F_2)^*)\)
What Would Tutte Do?
Chain groups

Definition: $R$ ring, $E$ finite set. Chain group is
\[ C \subseteq R^E \]
such that, for $c, d \in C$ and $r \in R$

- $0 \in C$
- $c + d \in C$
- $rc \in C$

Definition: Support of a chain $c$:
\[ \|c\| := \{ e \in E : c_e \neq 0 \} \]

Definition: Elementary chain: $c \neq 0$, inclusionwise minimal support.
Chain groups

Definition: Skew partial field $\mathbb{P} = (R, G)$

- $R$ ring
- $G \subseteq R^*$ group
- $-1 \in G$

Definition: $G$-primitive chain: $c \in (G \cup \{0\})^E$.

Definition: Chain group is $\mathbb{P}$-chain group if, for all $c \in C$ elementary,

$$c = rd$$

where $r \in R$ and $d \in C$ is $G$-primitive.
Chain groups

Theorem (Pendavingh, vZ 2009): For a \( P \)-chain group \( C \), define

\[
C^* := \{ \|c\| : c \in C, \text{elementary} \}
\]

Then \( C^* \) is set of cocircuits of a matroid, \( M(C) \).

(Co)circuit axioms

\( C^* \) is set of cocircuits of a matroid if and only if

- \( \emptyset \notin C^* \)
- \( C, D \in C^* \) and \( C \subseteq D \) then \( C = D \)
- \( C, D \in C^* \), \( C \neq D \), \( e \in C \cap D \), then \( (C \cup D) - e \) contains a cocircuit
**Homomorphisms**

**Definition.**
A *partial field homomorphism* \( \varphi : \mathbb{P}_1 \to \mathbb{P}_2 \) is

- Ring homomorphism \( R_1 \to R_2 \)
- Such that \( \varphi(G_1) \subseteq G_2 \)

**Theorem.**
If \( C \) is a \( \mathbb{P}_1 \)-chain group then \( \varphi(C) \) is a \( \mathbb{P}_2 \)-chain group.

**Examples:**
- \((\mathbb{F}_1 \times \mathbb{F}_2, (\mathbb{F}_1 \times \mathbb{F}_2)^*) \to (\mathbb{F}_1, \mathbb{F}_1^*)\)
- \((\mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle) \to \text{GF}(5)\) with \( \frac{1}{2} \mapsto 3 \)
Proof of Whittle’s classification

Theorem (Whittle 1997).
Equivalent for a matroid $M$:

- $M$ representable over all fields with characteristic $\neq 2$
- $M$ representable over $\text{GF}(3)$ and $\text{GF}(5)$
- $M$ has dyadic representation

Need to “reverse”

$$\varphi : \left( \mathbb{Z}[\frac{1}{2}], \langle -1, 2 \rangle \right) \to (\text{GF}(3) \times \text{GF}(5), (\text{GF}(3) \times \text{GF}(5))^*)$$
Lift Theorem

Theorem (Pendavingh, vZ 2010).
Let $P_1, P_2$ be partial fields, $\varphi : P_1 \to P_2$ homomorphism. If bijection between representations over $P_1$ and $P_2$ of

\[
\begin{cases}
\{, , , \}
\end{cases}
\]

Then $\mathcal{M}(P_1) = \mathcal{M}(P_2)$.

Proof generalizes Gerards’ (1989) proof of excluded minors for regular matroids.
Two conjectures

Conjecture
Equivalent are:

- Representable over $GF(2^k)$ for $k \geq 2$
- Representable over

$$\bigcup_1^{(2)} = (GF(2)(\alpha), \{\alpha^k(1 + \alpha)^l\})$$

Conjecture
Equivalent are:

- Representable over $\mathbb{F}$ if $|\mathbb{F}| \geq 4$
- Representable over

$$\mathbb{P}_4 = (\mathbb{Q}(\alpha), \{\alpha^k(1 - \alpha)^l(1 + \alpha)^m(2 - \alpha)^n\})$$
Application: excluded minors for near-regular

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[ M : E(M) \rightarrow \{ \text{GF}(3), \text{GF}(4), \text{GF}(5), \text{GF}(7), \ldots \} \]
namely
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for $M : E(M) \rightarrow \{\text{GF}(3), \text{GF}(4), \text{GF}(5), \text{GF}(7), \ldots\}$
Near-regular matroids

Theorem (Hall, Mayhew, vZ 2009):
Exactly 10 excluded minors for

\[ M : E(M) \rightarrow \mathbb{F}_2 \cup _1 \]
Regular matroids

Definition.
Matrix $A$ over $\mathbb{R}$ is totally unimodular if each nonsingular square submatrix $D$ has

$$|\det(D)| = 1.$$ 

Matrix Tree Theorem (Kirchhoff 1847).

$$\det(AA^T) = \# \{B : B \text{ basis of } M[A]\}.$$
Sixth-roots-of-unity matroids

Definition.
Matrix $A$ over $\mathbb{C}$ is \textit{complex unimodular} if each nonsingular square submatrix $D$ has

$$|\det(D)| = 1.$$ 

Theorem (Choe, Oxley, Sokal, Wagner 2004).
Complex-unimodular matroid is sixth-roots-of-unity: has representation with all entries in $\{0, 1, \zeta, \zeta^2, \ldots, \zeta^5\}$. 

Theorem (Choe, Oxley, Sokal, Wagner 2004).

$$\det(AA^+) = \#\{B : B \text{ basis of } M[A]\}.$$
Quaternionic unimodular matroids

Definition.
Matrix $A$ over $\mathbb{H}$ is quaternionic unimodular if each nonsingular square submatrix $D$ has

$$\delta(D) = 1.$$  

Theorem (Pendavingh, vZ 2013).

$$\delta(AA^\dagger) = \# \{B : B \text{ basis of } M[A]\}.$$
**Cauchy-binet extended**

**Theorem (Pendavingh, vZ 2013)**  
Let $A$ be $r \times s$ matrix over $\mathbb{H}$. Then  
\[
\delta(\text{AA}^\dagger) = \sum_{|X|=r} \delta(A_X)\delta(A_X^\dagger)
\]

where  
\[
\delta(D) := \sqrt{|\det(z_2(\varphi(D)))|}
\]
Basis counting, extended

$$\delta(AA^\dagger) = \# \{ B \text{ basis of } M[A] \}$$

$$P_A := A^\dagger(AA^\dagger)^{-1}A$$

$$\delta(P_A[F,F]) = \frac{\# \{ B \text{ basis, } F \subseteq B \}}{\# \{ B \text{ basis} \}}$$
Some open problems

Let $\mathbb{P}$ be skew partial field.

- Are $\mathbb{P}$-representable matroids algebraic?
- Does Ingleton’s Inequality hold?
- Are there Q.U. matroids not representable over a commutative field?
- Can we get all Q.U. matroids with just a finite subgroup of $\{x \in \mathbb{H} : \|x\| = 1\}$?
- Do Q.U. matroids have the half-plane property?
Multilinear representations

Definition.
A multilinear representation of $M$ is a map $A$: With each $e \in E(M)$ associate $k$-dim. vector space $A(e)$. For $S \subseteq E$:

$$\dim \bigoplus_{e \in S} A(e) = k \cdot r_M(S).$$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 2 & 0 & 6 & 0 & 6 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & -6 & 6 & -6 & 6 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 3 & 0 & 6 & 6 & 3 & 0 & 6 & 6 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 3 & -6 & 0 & -3 & 3 & -6 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]
More than a skew field

Theorem (Pendavingh, vZ 2013).
Let $M := R_9 \oplus Q_3(G)$. Then $M$ is representable over a skew partial field, but over no skew field.
Universal partial fields

Theorem.
Let $M$ be a matroid. There exist $P_M, A_M$ such that, whenever $M = M(A)$ over $F$, then

$$A \approx \varphi(A_M).$$

What are $P_M, A_M$?

• Formal definition using bracket ring
• Intuitive: matrix with variables
• Can (in principle) be computed with Groebner basis!

Problem.
Develop a theory of universal skew partial fields
**Templates**

**Theorem (Grace, in preparation).**
Let $\mathcal{M}$ be class of quaternary matroids representable over field of all characteristics. There exist $k, l$ such that the $k$-connected matroids of size $\geq l$ in $\mathcal{M}$ are in one of the following classes:

- representable over *all* fields of size $\geq 4$;
- representable over $\text{GF}(4)$ and $\text{GF}(q)$ for $q \geq 7$;
- golden ratio.

**Note:**

**Theorem (Whittle 2005).**
There exist infinitely many classes of matroids representable over $\text{GF}(4)$ and $\text{GF}(q)$. 
The Structure of Highly Connected Matroids

Geelen, Gerards, Whittle announced proof of the following:

**Hypothesis.**
\( \mathcal{M} \) proper minor-closed class of binary matroids. If \( M \in \mathcal{M} \) is sufficiently large and has sufficiently high branch-width, then \( M \) has a tree-decomposition, the parts of which correspond to mild modifications of graphic matroids and their duals.

**Perturbation:** add low-rank matrix to representation. Matroidal view: small number of lifts and projections.
More detail: templates

**Definition (Geelen, Gerards, Whittle).**

*Binary Frame Template* is tuple \( \Phi = (C, D, Y_0, Y_1, A_1, \Delta, \Lambda) \).

Matrix \( A \) respects \( \Phi \) if of the form

\[
\begin{array}{ccc}
D & columns \text{ from } \Lambda & Z \\
& & 0 \\
& Graphic \text{ columns} & unit \text{ columns} \\
& & rows \text{ from } \Delta \\
\end{array}
\]

\[
\begin{array}{ccc}
& & Y_0 \\
& & Y_1 \\
& & A_1 \\
\end{array}
\]
Slides, preprints at
http://www.math.lsu.edu/~svanzwam/

The End