Working with Templates

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Part I
Structure of minor-closed classes

Graph Minors Structure Theorem

Theorem (Robertson and Seymour 2003).
Let $\mathcal{G}$ be proper minor-closed class of graphs. Each $G \in \mathcal{G}$ admits a *tree-decomposition*, whose parts are *almost embeddable in a surface*.

Consequences:

- No infinite antichains of graphs;
- $\mathcal{G}$ has finite set of excluded minors;
- Algorithms.

Matroid minors: the blueprint

Theorem (Seymour 1980).
Let M be a *regular* matroid. Then M can be constructed from graphic matroids, cographic matroids, and R_{10} through 1-, 2-, 3-sums.

Highly connected regular matroids are:

- Graphic matroids
- Cographic matroids

What can happen for other classes?

Matroid Minors Structure Theorem

Hypothesis (Geelen, Gerards, Whittle; rough idea).
$\mathcal{M}$ proper minor-closed class of $\mathbb{F}$-representable matroids. If $M \in \mathcal{M}$ has sufficiently high branch-width, then M has a tree-decomposition, the parts of which are *mild modifications* of

- representable over a subfield of $\mathbb{F}$; or
- frame matroids; or
- duals of frame matroids.

Need: lots of definitions, 15 years of hard work by GGW.

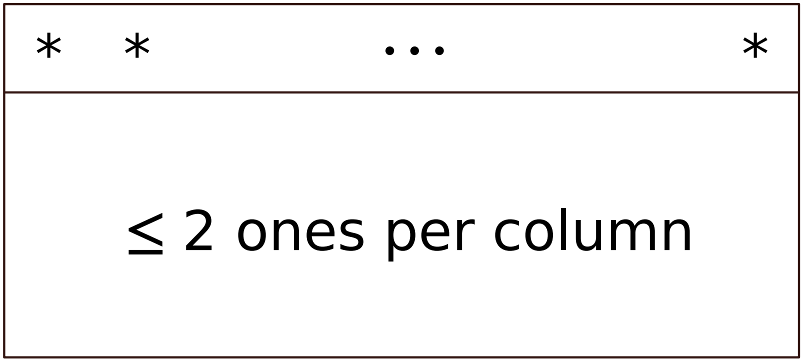
Constructions

Frame matroids

≤ 2 ones per column

Constructions

Even-cycle matroids (binary)



Constructions

Grafts (binary)

$$\left[\begin{array}{c|c} & * \\ \leq 2 \text{ ones per column} & \vdots \\ & * \end{array}\right]$$

Close under minors: duals of *even-cut matroids* (Guenin, Pivotto, Wollan).

Constructions

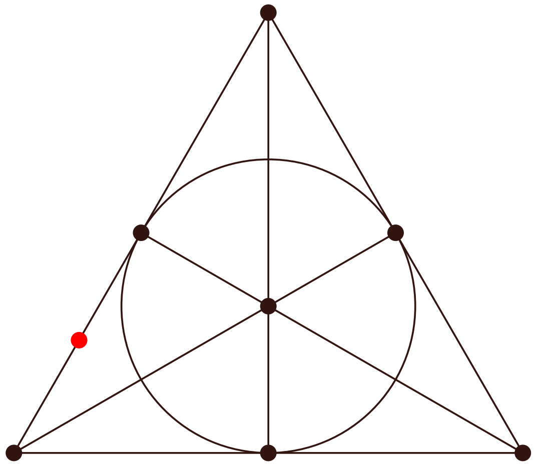
Almost representable over a subfield

A matroid is GF(q)-regular if it is representable over GF(q^t) for all $t \geq 2$.

Theorem (Nelson, vZ 2015)

If M is highly connected and has a large PG(t, q) minor, then equivalent:

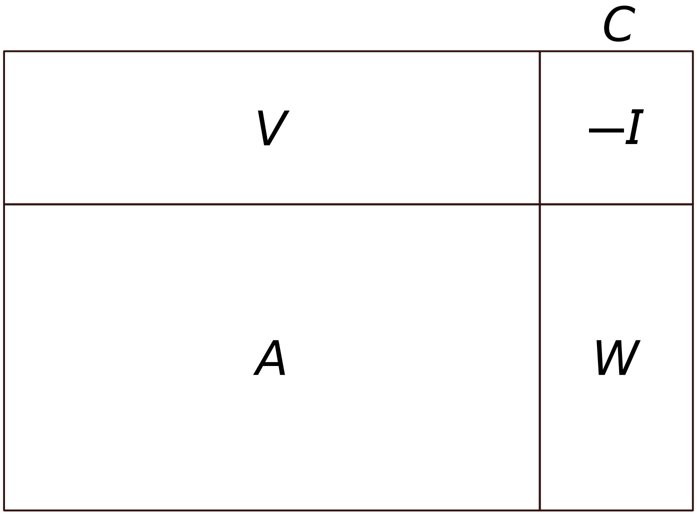
- M is GF(q)-regular;
- M is representable over GF(q^2) and GF(q^t) for some $t \geq 3$;
- M is a restriction of $\widehat{\mathrm{PG}}(r-1, q)$ or $\overline{\mathrm{PG}}(r-1, q)$.



Perturbations

Definition.

A *rank-*$(\leq t)$ *perturbation* of $M = M[A]$ is the matroid $M[A + P]$, where P has matrix rank $\leq t$.



Matroid view: $\leq t$ *lifts* and *projections*.

Matroid Minors Structure Theorem

Hypothesis (Geelen, Gerards, Whittle 2015). Let $\mathcal{M}$ be proper minor-closed class of matroids representable over GF(q). There exist k, t, l such that each vertically k-connected $M \in \mathcal{M}$ of size $\geq l$ is

- rank-($\leq t$) perturbation of frame matroid; or
- dual of rank-($\leq t$) perturbation of frame matroid; or
- rank-($\leq t$) perturbation of matroid representable over a subfield.

Application: Linear Error-correcting Codes

- Binary linear code C is k-dimensional sub*space* of $\mathrm{GF}(2)^n$.
- Notation: $[n, k, d]$ linear code.

Asymptotically good codes

- Family $C_1, C_2, \ldots$ of linear codes with parameters $[n_i, k_i, d_i]$ is *asymptotically good* if, for some $\varepsilon > 0$:

(*i*) *Growing size:* $n_i \to \infty$ as $i \to \infty$

(*ii*) *Constant rate:* $k_i/n_i \geq \varepsilon$

(*iii*) *Growing minimum distance:* $d_i/n_i \geq \varepsilon$

Theorem. Asymptotically good codes exist.

Asymptotically good codes: structure?

Operations on a code:

- **Puncturing:** $C \setminus i$, remove ith coordinate from each word
- **Shortening:** C/i, take $\{c \in C : c_i = 0\}$, then remove ith coordinate.

Theorem (Nelson, vZ 2015). Let $\mathcal{M}$ be a class of binary linear codes closed under puncturing, shortening. Assuming Hypothesis, if $\mathcal{M}$ contains an asymptotically good sequence, then $\mathcal{M}$ contains *all* codes.

Part II
A Speed Bump

Trouble

Theorem? (Geelen, Gerards, Whittle 2015). Let $\mathcal{M}$ be proper minor-closed class of matroids representable over GF(q). There exist k, t, l such that each vertically k-connected $M \in \mathcal{M}$ of size $\geq l$ is

- rank-($\leq t$) perturbation of frame matroid; or
- dual of rank-($\leq t$) perturbation of frame matroid; or
- rank-($\leq t$) perturbation of matroid represented over a subfield.

A counterexample

Let $\mathcal{D}$ be class of *dyadic* matroids (i.e. representable over GF(3) and GF(5)).

Theorem (Grace, vZ 2017+).

For each k, t, l there exists a vertically k-connected dyadic matroid M on $\geq l$ elements, such that **NO** rank-**(**$\leq t$) perturbation is a represented frame matroid or the dual of a represented frame matroid.

Consolation:

- Vertical k-connectivity and cographic don't mix;
- Most results saved by going to cyclic k-connectivity when "almost dual of frame";
- Everything should hold when the word "vertically" is struck out.

Matroid Minors Structure Theorem

Hypothesis (Geelen, Gerards, Whittle 2015). Let $\mathcal{M}$ be proper minor-closed class of matroids representable over GF(q). There exist k, t, l such that each k-connected $M \in \mathcal{M}$ of size $\geq l$ is

- rank-($\leq t$) perturbation of frame matroid; or
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- rank-($\leq t$) perturbation of matroid representable over a subfield.

Part III
More detail: templates

Frame templates

Definition.
$\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$.

		Z	Y_1 Y_0 C
X	columns from Λ	0	A_1
	≤ 2 ones per column	unit columns	rows from Δ

Definition.
$\mathcal{M}(\Phi)$ is set of matroids *conforming to* template Φ.

Frame Template Hypothesis

Hypothesis (Geelen, Gerards, Whittle 2015)

$\mathcal{M}$ minor-closed class of GF(q)-representable matroids, m integer. $\exists k, \Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ s.t.

- $\forall i : \mathcal{M}(\Phi_i) \subseteq \mathcal{M}$;
- $\forall j : \mathcal{M}^*(\Psi_j) \subseteq \mathcal{M}$;
- If $M \in \mathcal{M}$ is simple, k-connected, $\geq 2k$ elements, no PG(m, p)-minor, then
  - $M \in \mathcal{M}(\Phi_1) \cup \cdots \cup \mathcal{M}(\Phi_s)$ or
  - $M^* \in \mathcal{M}(\Psi_1) \cup \cdots \cup \mathcal{M}(\Psi_s)$.

Refine connectivity:

- Vertically k-connected: M has large $M(K_n)$ minor;
- Cyclically k-connected: M has large $M^*(K_n)$ minor.

Creating an order on templates

Theorem (Grace, vZ 2017).
For every binary frame template Φ, one of the following holds:

- Φ is trivial;
- Φ reduces to Φ_C, Φ_X, Φ_{CX}, Φ_{Y_0}, or Φ_{Y_1};
- For some k, l no simple, vertically k-connected matroid of size $\geq l$ conforms or coconforms to Φ.

		Z	Y_1 Y_0 C
X	columns from Λ	0	A_1
	≤ 2 ones per column	unit columns	rows from Δ

Creating an order on templates

Φ_0 :

$\leq$ 2nonzeroes per col

Φ_X :

$\underline{x}$
$\leq$ 2nonzeroes per col

Φ_C / Φ_{Y_0}

$\leq$ 2nonzeroes per col	$\underline{y}$

Φ_{CX}

$\underline{x}$	1
$\leq$ 2nonzeroes per col	$\underline{y}$

Φ_{Y_1}

	$\begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{matrix}$	$\begin{matrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{matrix}$	$\begin{matrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$
$\leq$ 2nonzeroes per col	I	I	I	0

Application: 1-flowing matroids

Conjecture (Seymour 1981).
The excluded minors for 1-flowing matroids are $U_{2,4}$, $\mathrm{AG}(3,2)$, T_{11}, T_{11}^*.

Theorem (Grace, vZ 2017).
The template list for 1-flowing matroids is $\{\Phi_0\}$.

Corollary (Grace, vZ 2017).
Subject to Template Covering Hypothesis, a counterexample to Seymour's 1-Flowing Conjecture has low-order separation or small size.

Approach

$$\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda).$$

For class $\mathcal{M}$: explicitly find full list of templates Φ such that $\mathcal{M}(\Phi) \subseteq \mathcal{M}$.
To rule out a potential template:

- Show $\mathcal{M}(\Phi) \subseteq \mathcal{M}(\Phi')$ with Φ' already in list; or
- Show the matroids conforming to the template are not highly connected; or
- Find certificate placing it outside $\mathcal{M}$.
  - Typically, try to build an *excluded minor for* $\mathcal{M}$ using Φ.

Note: This procedure yields theorems, independent of hypotheses!

Tools

- Reduction operations
- Refined template:

		Z	Y_1		Y_0	C
X_1	0	0	I	$*$	$*$	0
X_0	columns from $\Lambda[X_0]$		0	$*$	$*$	$*$
	Γ-frame	unit columns	rows from Δ			

(Extra conditions on Λ and Δ too).

Slides, articles at
http://www.math.lsu.edu/~svanzwam/

Goodbye