Presentations and Extensions of Transversal Matroids

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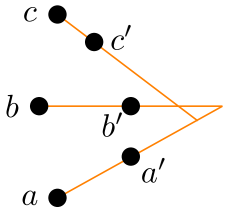
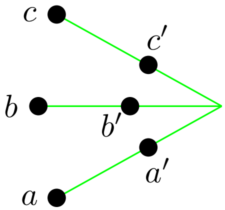
The motivation, by analogy

A matroid M can have inequivalent representations, so a matrix representation A can have extraneous information that limits which extensions of M can be represented by adding a column to A.



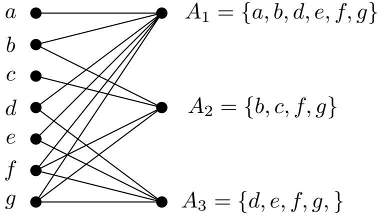


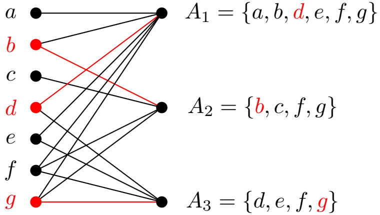
Transversals matroid often have many presentations.

How do the presentations relate to (e.g., limit) the transversal extensions?

A brief review of transversal matroids

A set system, $\mathcal{A} = (A_i : i \in [r])$, is a multiset of sets.
E.g., $A_1 = \{a, b, d, e, f, g\}$, $A_2 = \{b, c, f, g\}$, $A_3 = \{d, e, f, g, \}$.





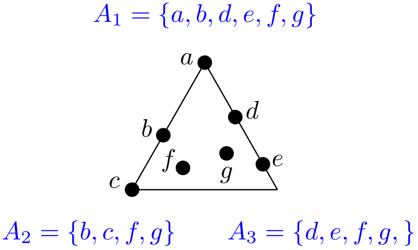
The independent sets of the transversal matroid $M[\mathcal{A}]$ are the partial transversals of $\mathcal{A}$.

The set system $\mathcal{A}$ is a presentation of $M[\mathcal{A}]$.

A brief review of transversal matroids

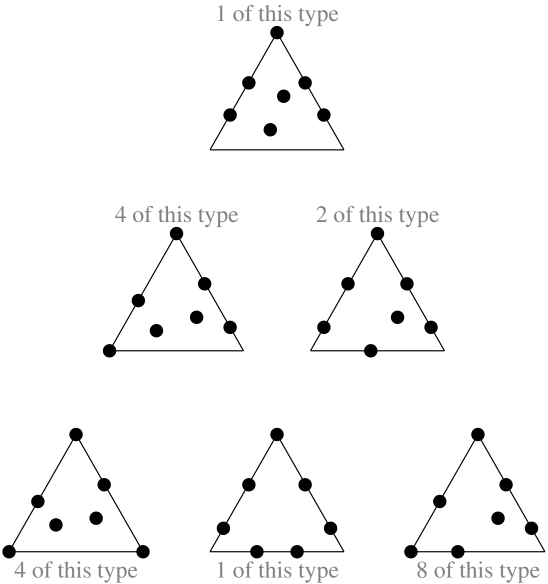
$$A_1 = \{a, b, d, e, f, g\}, \quad A_2 = \{b, c, f, g\}, \quad A_3 = \{d, e, f, g, \}$$

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{pmatrix} * & * & 0 & * & * & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & * \end{pmatrix} \end{array}$$

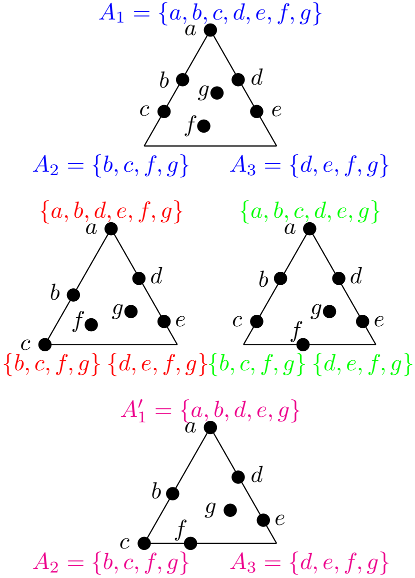


We can replace $*$s with positive reals so that (i) determinants that aren't forced to be zero for generic reasons are nonzero and (ii) each non-zero column sums to 1.

The multitude of presentations



The order on presentations



A partial order on the presentations of M:

$(A_1, \ldots, A_r) \leq (A_1', \ldots, A_r')$ if, up to re-indexing the sets, $A_i \subseteq A_i'$ for all $i \in [r]$.

A transversal matroid has a unique maximal presentation.

(Mason, 1971.)

Typically there are many minimal presentations.

Extensions of a presentation of a transversal matroid

We consider only
(a) presentations of M with $r = r(M)$ sets, and
(b) single-element transversal extensions of M of rank r.

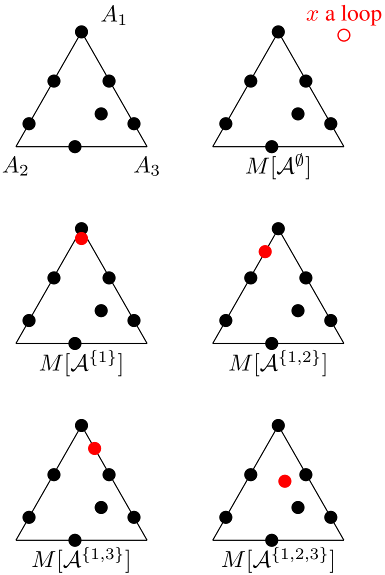
Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of M.

For $x \notin E(M)$ and $J \subseteq [r]$, set $\mathcal{A}^J = (A_i^J : i \in [r])$ where

$$A_i^J = \begin{cases} A_i \cup x, & \text{if } i \in J, \\ A_i, & \text{otherwise.} \end{cases}$$

The extension $\mathcal{A}^J$ of $\mathcal{A}$ yields the extension $M[\mathcal{A}^J]$ of M.

Examples of extensions

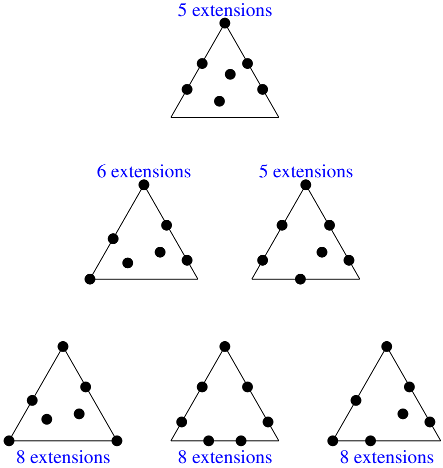


$$M[\mathcal{A}^{\{2\}}] = M[\mathcal{A}^{\{1,2\}}]$$

$$M[\mathcal{A}^{\{3\}}] = M[\mathcal{A}^{\{1,3\}}]$$

$$M[\mathcal{A}^{\{2,3\}}] = M[\mathcal{A}^{\{1,2,3\}}]$$

New results on the number of extensions of M from one presentation



Let $\mathcal{T}_{\mathcal{A}}$ be the set of transversal extensions of M obtained by extending $\mathcal{A}$.

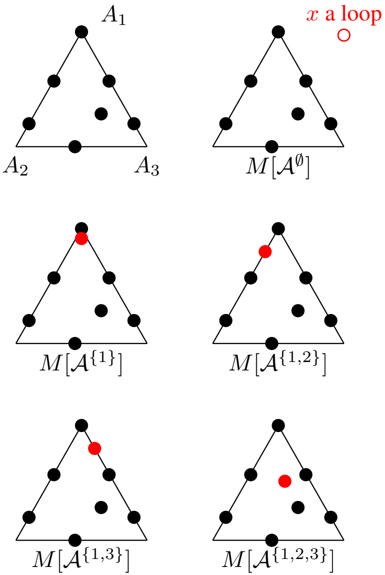
Theorem

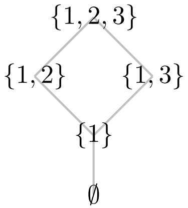
Let $r = r(M)$.

$\mathcal{A}$ is minimal iff $|\mathcal{T}_{\mathcal{A}}| = 2^r$; otherwise $|\mathcal{T}_{\mathcal{A}}| \leq \frac{3}{4} \cdot 2^r$.

If $\mathcal{A}$ neither is minimal nor covers a minimal presentation, then $|\mathcal{T}_{\mathcal{A}}| \leq \frac{5}{8} \cdot 2^r$. (2014)

A lattice





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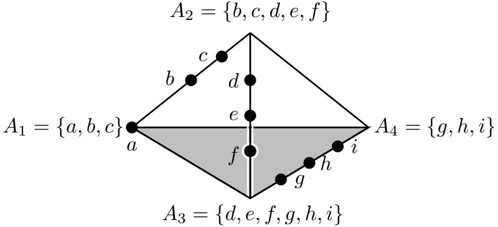
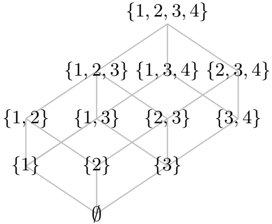
A second example of such a lattice





A closure operator on a set S is a map $\sigma : 2^S \to 2^S$ for which

1. $X \subseteq \sigma(X)$ for all $X \subseteq S$,
2. if $X \subseteq Y \subseteq S$, then $\sigma(X) \subseteq \sigma(Y)$, and
3. $\sigma(\sigma(X)) = \sigma(X)$ for all $X \subseteq S$.

A σ-closed set is a subset X of S with $\sigma(X) = X$.

The σ-closed sets, ordered by inclusion, form a lattice;
the lattice operations are $X \vee Y = \sigma(X \cup Y)$ and $X \wedge Y = X \cap Y$.

Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of M, where $r = r(M)$.

Lemma

For $e \in E(M) - A_i$, $\quad (A_1, A_2, \ldots, A_{i-1}, A_i \cup e, A_{i+1}, \ldots, A_r)$ *is a presentation of M iff e is a coloop of $M \backslash A_i$.* (Bondy & Welsh, 1971)

So, for $J \subseteq [r]$, there is a maximum $K \subseteq [r]$ with $M[\mathcal{A}^J] = M[\mathcal{A}^K]$.

Define $\sigma_{\mathcal{A}} : 2^{[r]} \to 2^{[r]}$ by $\sigma_{\mathcal{A}}(J) = K$.

Theorem

The map $\sigma_{\mathcal{A}}$ is a closure operator on $[r]$. The join of I and J in the lattice $L_{\mathcal{A}}$ of $\sigma_{\mathcal{A}}$-closed sets is $I \cup J$, so $L_{\mathcal{A}}$ is distributive. If $\mathcal{B} = (B_i : i \in [r])$ is a presentation of M with $A_i \subseteq B_i$ for all i, then $L_{\mathcal{B}}$ is a sublattice of $L_{\mathcal{A}}$. (2014)

Order $\mathcal{T}_{\mathcal{A}}$, the set of extensions of M obtained by extending $\mathcal{A}$, by the weak order.

For matroids M and N on E, we have $M \leq_w N$ in the weak order if $r_M(X) \leq r_N(X)$ for all $X \subseteq E$; equivalently, every independent set of M is independent in N.

Theorem

The map $J \mapsto M[\mathcal{A}^J]$ is an isomorphism from $L_{\mathcal{A}}$ onto $\mathcal{T}_{\mathcal{A}}$, so $\mathcal{T}_{\mathcal{A}}$ is a distributive lattice. *(2014)*

Theorem

If $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}_{\mathcal{A}}$.

If $A_i \subseteq B_i$ for all $i \in [r]$, and if $J \subseteq [r]$, then $M[\mathcal{A}^{\sigma_{\mathcal{B}}(J)}] = M[\mathcal{B}^J]$.

(2014)

Corollary

For any single-element transversal extension M' of M, there is a minimal presentation $\mathcal{A}$ of M and a set J with $M' = M[\mathcal{A}^J]$. *(2013)*

Theorem

Let $r = r(M)$. A presentation $\mathcal{A}$ of M is minimal iff $|\mathcal{T}_{\mathcal{A}}| = 2^r$.

(2013)

Lemma

The presentation $\mathcal{A}$ is minimal iff each set in $\mathcal{A}$ is a cocircuit of M.

(Las Vergnas, 1971; Bondy and Welsh, 1971)

If $\mathcal{A}$ is not minimal, then $r(M \backslash A_i) < r - 1$ for some $A_i \in \mathcal{A}$.

Set $J = [r] - i$.

So x is a coloop of $M[\mathcal{A}^J] \backslash A_i$, so $M[\mathcal{A}^J] = M[\mathcal{A}^{J \cup i}]$, so $|\mathcal{T}_{\mathcal{A}}| < 2^r$.

Theorem

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(2013)

Lemma

The presentation $\mathcal{A}$ is minimal iff each set in $\mathcal{A}$ is a cocircuit of M.

(Las Vergnas, 1971; Bondy and Welsh, 1971)

If $\mathcal{A}$ is minimal and $i \in [r]$, then $r(M \backslash A_i) = r - 1$.

If $J \subseteq [r]$ and $i \notin J$, then x is not a coloop of $M[\mathcal{A}^J] \backslash A_i$ since $M[\mathcal{A}^J] \backslash A_i$ extends $M \backslash A_i$, so J is closed. Thus, $L_{\mathcal{A}} = 2^{[r]}$.

Theorem

If $\mathcal{A}$ is not minimal, then $|\mathcal{T}_{\mathcal{A}}| = |L_{\mathcal{A}}| \leq \frac{3}{4} \cdot 2^r$. (2014)

This follows by finding the maximal proper sublattices of $2^{[r]}$ that contain $\emptyset$ and $[r]$, which are

$$L_{ij} = \{X \subseteq [r] : X \cap \{i, j\} \neq \{i\}\}$$

for any distinct i, j in $[r]$; also, $|L_{ij}| = \frac{3}{4} \cdot 2^r$.

Finding the next-largest sublattices (two types), and showing that the number of extensions goes down, gives:

Theorem

If $\mathcal{A}$ neither is minimal nor covers a minimal presentation, then $|\mathcal{T}_{\mathcal{A}}| = |L_{\mathcal{A}}| \leq \frac{5}{8} \cdot 2^r$. (2014)

Theorem

Let L be a sublattice of $2^{[r]}$ (so $I \cap J \in L$ and $I \cup J \in L$ for all $I, J \in L$) with both $\emptyset$ and $[r]$ in L.

There is a transversal matroid M of rank r with $L = L_{\mathcal{A}}$, where $\mathcal{A}$ is the maximal presentation of M.

For each integer $n \geq r$, there is a presentation $\mathcal{A}$ of the uniform matroid $U_{r,n}$ on $[n]$ with $L = L_{\mathcal{A}}$. *(2014)*

How many extensions can be common to two presentations?

Theorem
If $\mathcal{A}$ and $\mathcal{B}$ are different presentations of M, then
$2 \leq |\mathcal{T}_\mathcal{A} \cap \mathcal{T}_\mathcal{B}| \leq \frac{3}{4} \cdot 2^r$. *(2014)*

Theorem
The set $L_{\mathcal{A},\mathcal{B}} = \{J \in L_\mathcal{A} : M[\mathcal{A}^J] = M[\mathcal{B}^K] \text{ for some } K \in L_\mathcal{B}\}$ *is a sublattice of* $L_\mathcal{A}$. *(2014)*

Closure under unions follows from:

Theorem
For all subsets J and K of $[r]$, the join of $M[\mathcal{A}^J]$ and $M[\mathcal{A}^K]$ in the lattice of all extensions of M is transversal and is $M[\mathcal{A}^{J \cup K}]$. *(2013)*

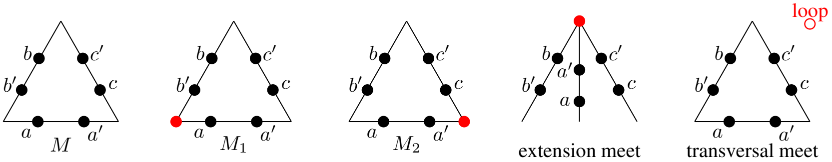
The corresponding statement for meets is false.

Under the weak order, the set of all single-element extensions of a matroid M is a lattice.

Question

Is the set of all single-element transversal extensions of M, ordered by the weak order, a lattice?

As we saw, $\mathcal{T}_{\mathcal{A}}$ is a lattice, but this question concerns the union of $\mathcal{T}_{\mathcal{A}}$ over all presentations $\mathcal{A}$ of M.



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