An algorithm for constructing a matroid's k-tree

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Joint work with Charles Semple

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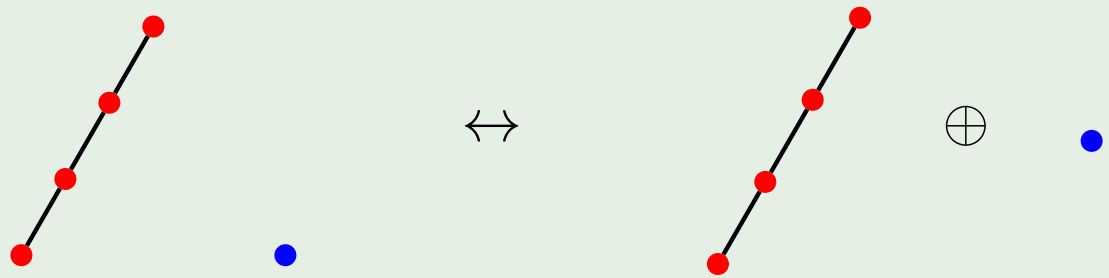
Matroid decompositions

- We can decompose a matroid into its (2-connected) components (Whitney, 1935)

Example



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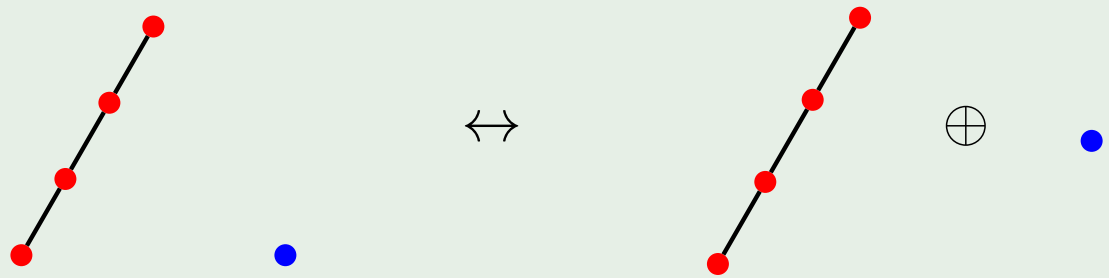

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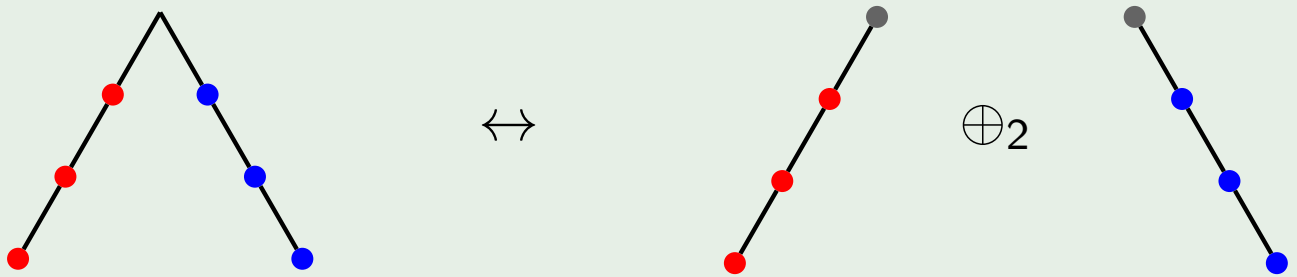

Matroid decompositions (2)

Theorem (Cunningham and Edmonds, 1980)

A 2-connected matroid has a tree decomposition in which every vertex label corresponds to a 3-connected matroid.

These results allow us to apply induction arguments

- Often have: "A matroid has property x if the 3-connected components of the 2-sum decomposition have property x"
- Contributed to initial research focus on 3-connected matroids

Can we decompose a 3-connected matroid into more highly-connected pieces?

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Crossing 3-separations

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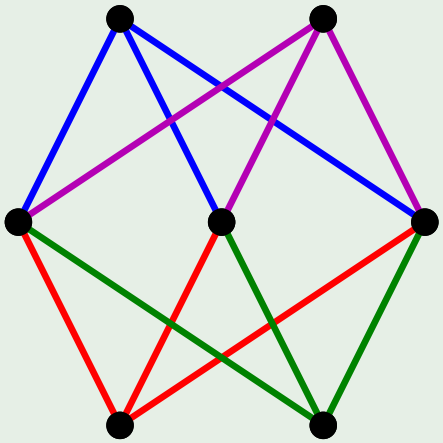

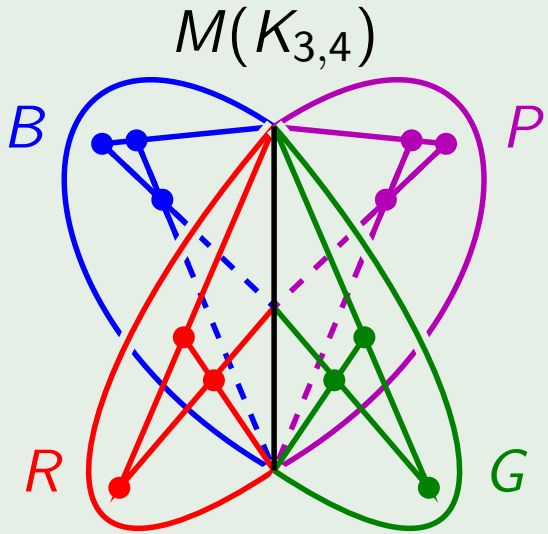

- 3-connected
- Crossing 3-separations: $(R \cup G, B \cup P)$, $(R \cup P, B \cup G)$ and, $(R \cup B, G \cup P)$.

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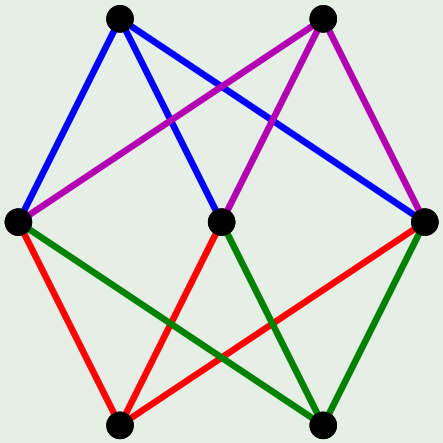

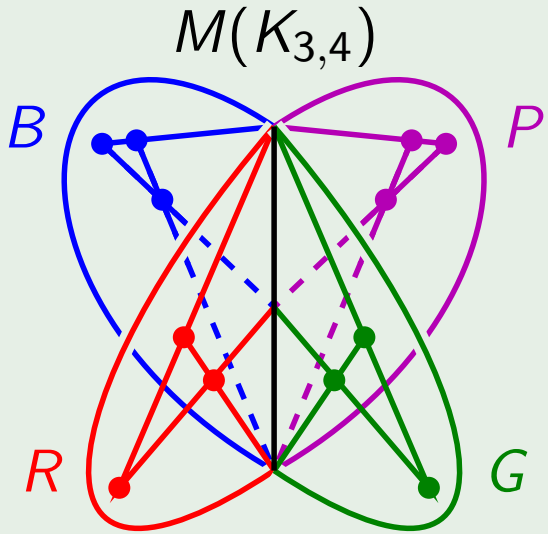

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Equivalent 3-separations

Theorem (Oxley, Semple, and Whittle, 2004)

For a 3-connected matroid M with $|E(M)| \geq 9$, there is a tree that displays, up to a natural equivalence, all the non-sequential 3-separations.

- Called a 3-tree

- X is fully closed if it is closed and coclosed
- $\mathrm{fcl}(X)$ is the intersection of all fully closed sets containing X
- (X, Y) is equivalent to (A, B) if $\{\mathrm{fcl}(X), \mathrm{fcl}(Y)\} = \{\mathrm{fcl}(A), \mathrm{fcl}(B)\}$

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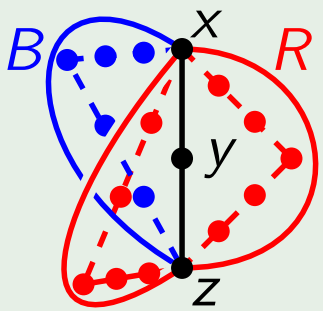

- $(R \cup \{x, y, z\}, B)$ and $(R, B \cup \{x, y, z\})$ are equivalent 3-separations

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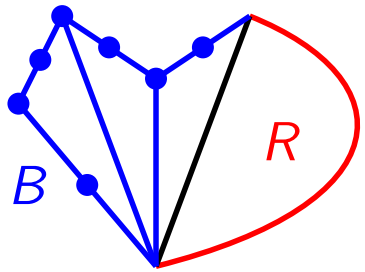
Equivalent 3-separations (2)

Theorem (Oxley, Semple, and Whittle, 2004)

For a 3-connected matroid M, there is a tree that displays, up to a natural equivalence, all the non-sequential 3-separations of M.

- (X, Y) is sequential if $\mathrm{fcl}(X) = E(M)$ or $\mathrm{fcl}(Y) = E(M)$
  - A sequential 3-separation (R, B):



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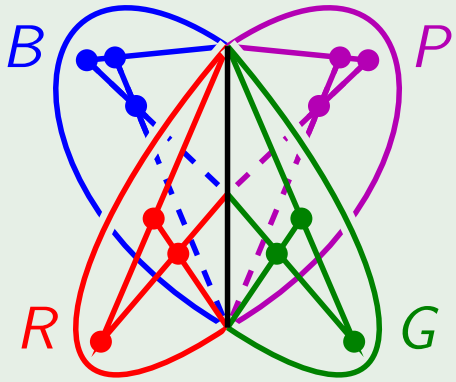
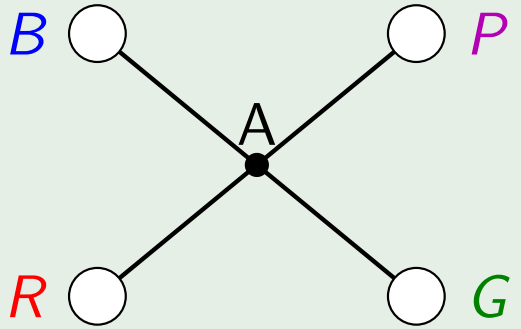


- The associated 3-tree is:



- Each edge corresponds to a 3-separation: $(R, B \cup P \cup G)$, $(B, P \cup G \cup R)$, $(P, G \cup R \cup B)$, and $(G, R \cup B \cup P)$
- (R, B, P, G) is a flower, and R, B, P, and G are the petals
  - A is a flower vertex
  - the others are bag vertices

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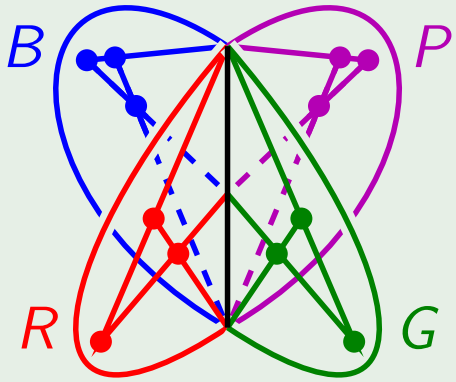

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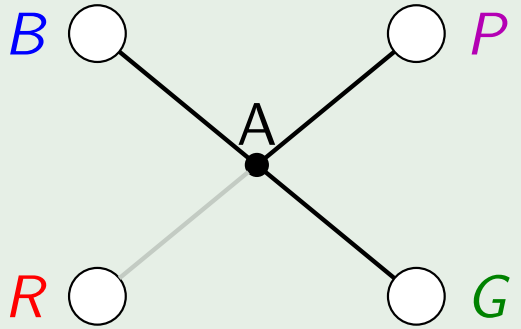

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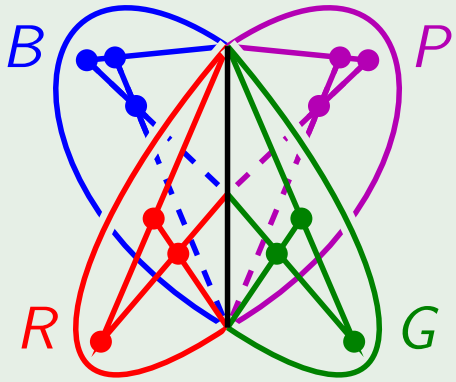

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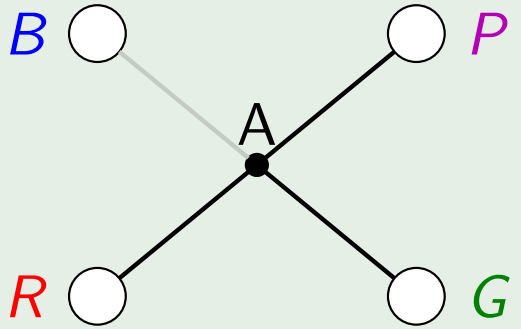

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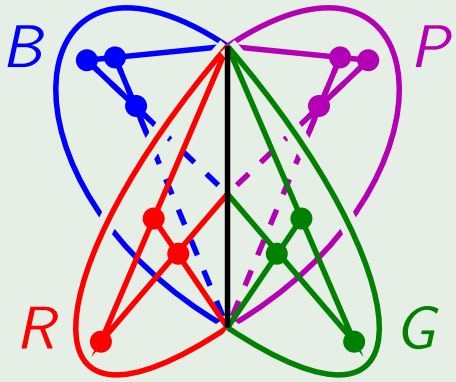

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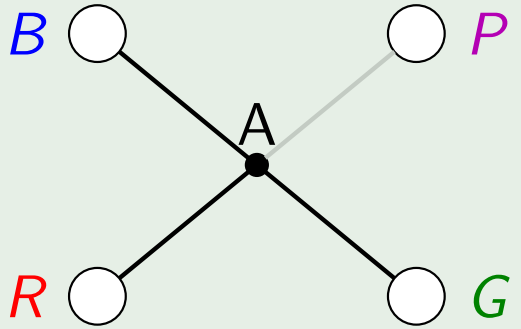

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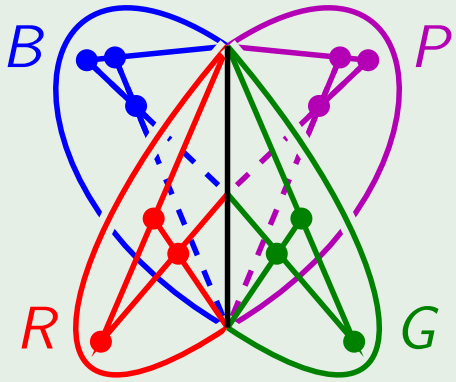

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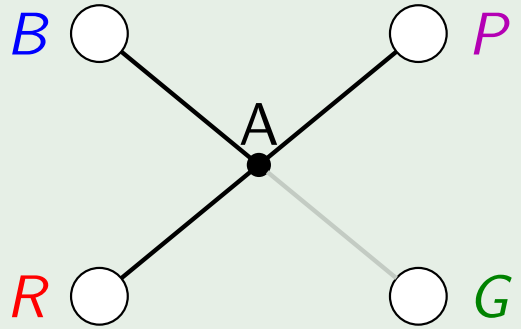

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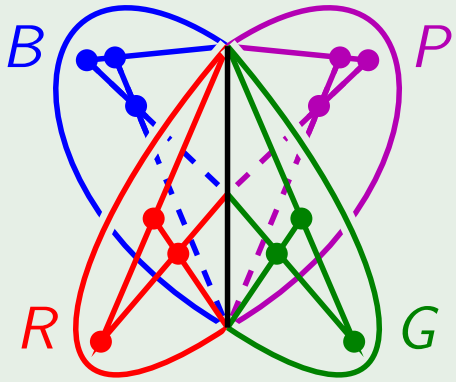

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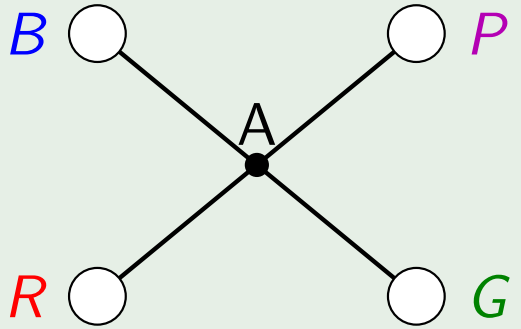

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- Flowers fall into two categories: anemones and daisies
- Any partition of petals in an anemone gives a 3-separation

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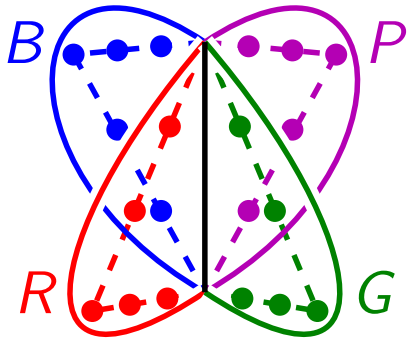

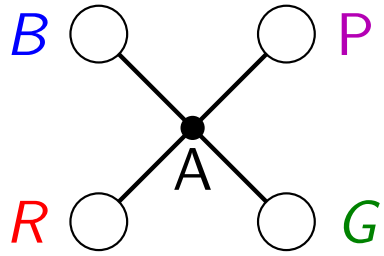

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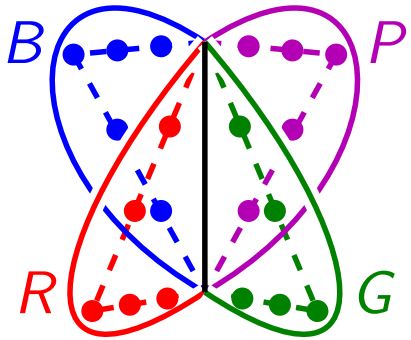

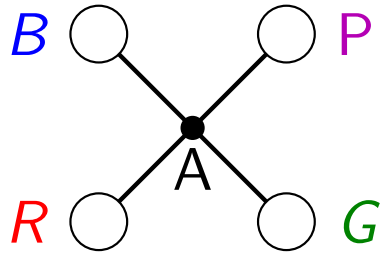

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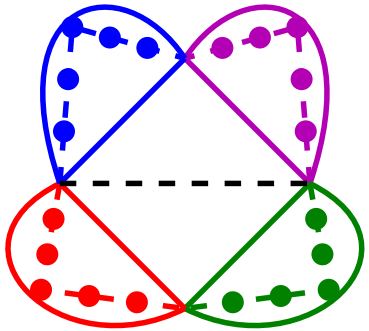

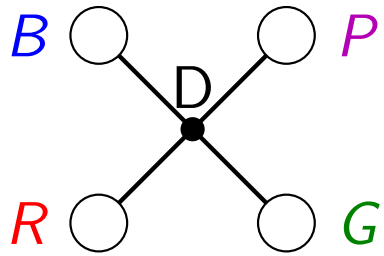

$(B \cup G, P \cup R)$ is not a 3-separation

Decompositions revisited

- A 3-connected representable matroid can be decomposed into sequentially 4-connected pieces and some special 3-connected matroids (Beavers 2006 / Chen and Xiang, 2012).
  - Associated tree describes how to rebuild the original from the parts
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Constructing 3-trees

For an arbitrary 3-connected matroid, can we construct its 3-tree in polynomial time?

- The proof in OSW2004 doesn't lend itself to a polynomial-time algorithm
  - We can find, in polynomial time, a non-sequential 3-separation (R, B) if one exists
  - Not known if, in polynomial time, we can identify if (R, B) is a tight maximal flower or whether it can be refined

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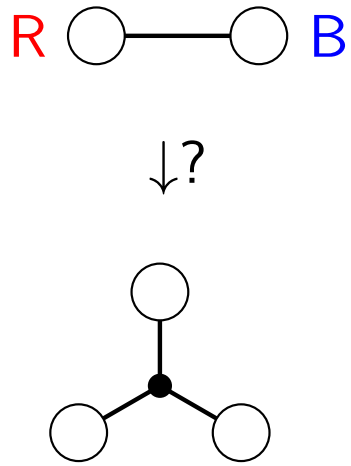

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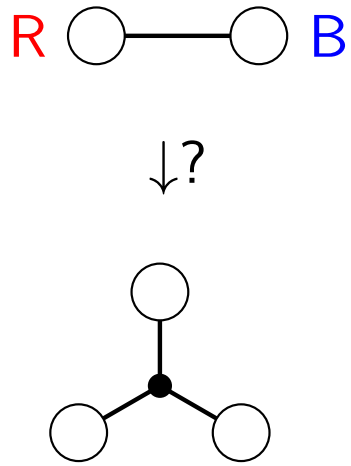

Finding non-sequential 3-separations

Theorem (Oxley and Semple, 2013)

A 3-tree for a 3-connected matroid can be constructed in polynomial time.

- Matroid specified by a rank oracle
- Need to find non-sequential 3-separations in polynomial time
  - The matroid intersection algorithm (Cunningham and Edmonds 1973) can be used to extend disjoint 3-element subsets X', Y' to a 3-separation (X, Y), if possible
  - Oxley and Semple showed that non-sequential 3-separations can be characterised by whether there are 3-element subsets X', Y' of X, Y (respectively) that are not contained in a maximal sequential set

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The 3-Tree algorithm

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A 3-tree for a 3-connected matroid can be constructed in polynomial time.

Outline of the 3-Tree algorithm:

1. ForwardSweep: Find the non-sequential 3-separations "in one direction" (a maximal 3-path)
2. BackwardSweep: Starting at the "unrooted" end, discover flower structure
3. While there is an unmarked bag B in the resulting tree
   - Fixing $E(M) - \pi(B)$ (at "rooted" end), call ForwardSweep, then BackwardSweep, to refine the structure of B, update the tree accordingly, then mark B

The 3-tree algorithm: an example

Example

ForwardSweep:

(X, Y)

$(X_1, X_2 \cup Y)$

$(X \cup Y_1, Y_2)$

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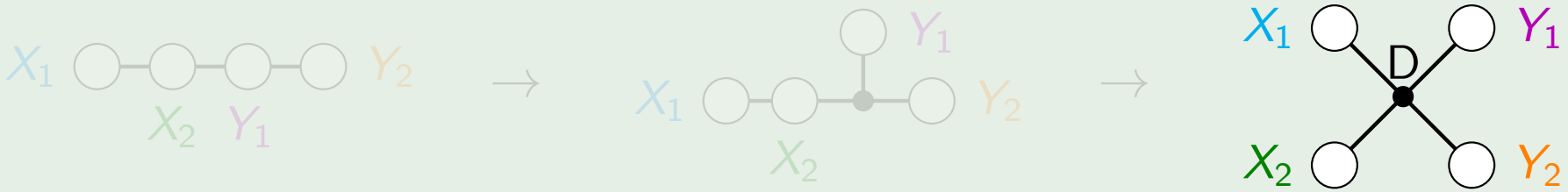

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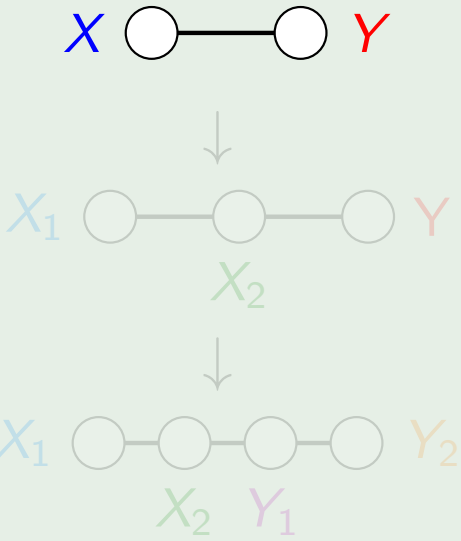

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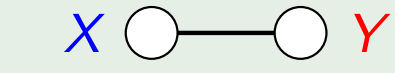

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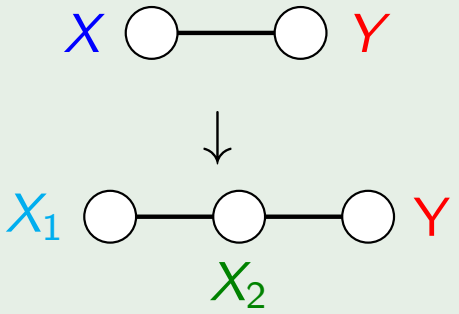

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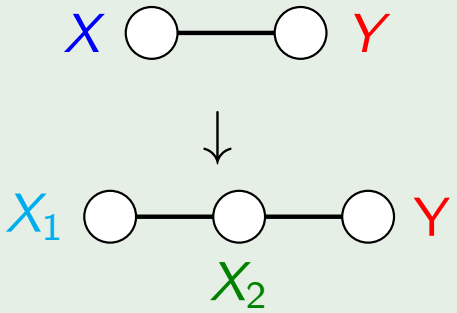

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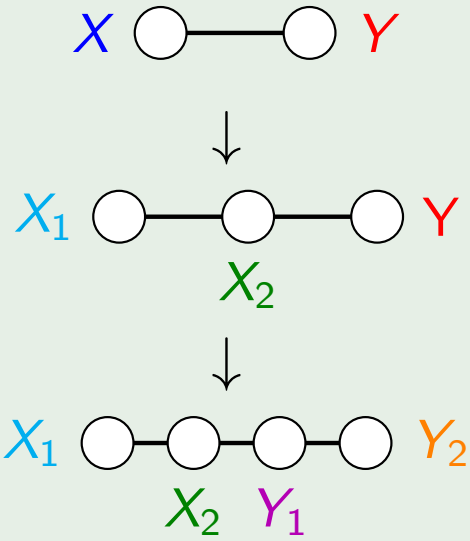

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$(X \cup Y_1, Y_2)$

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BackwardSweep:

The 3-tree algorithm: an example

Example

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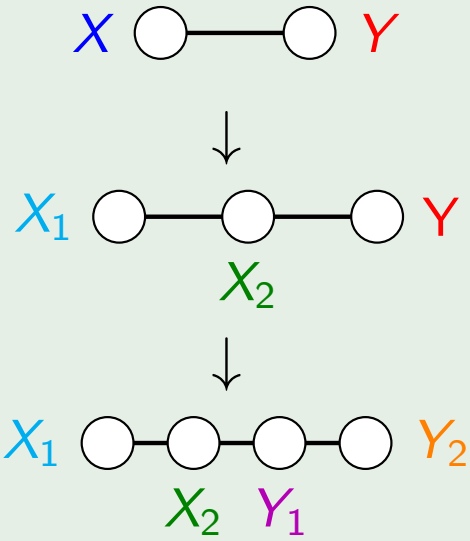

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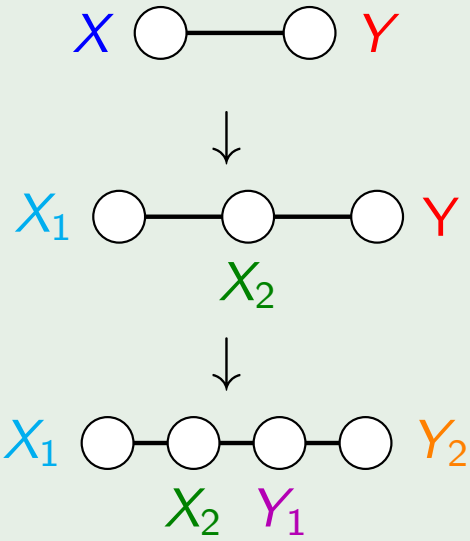

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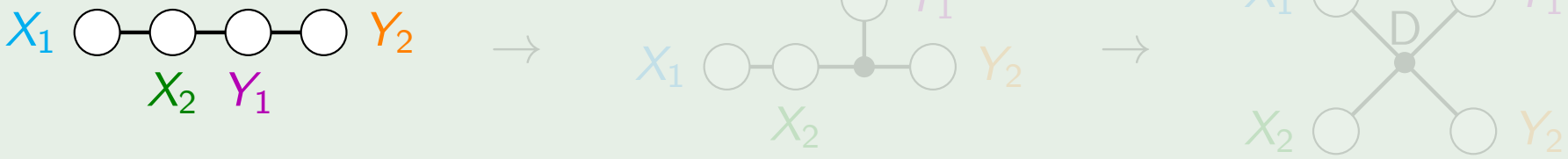

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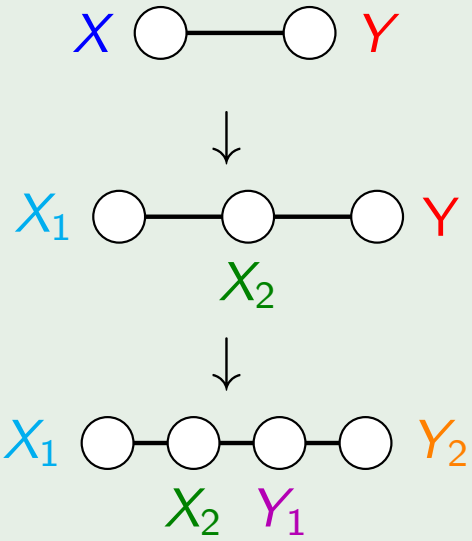

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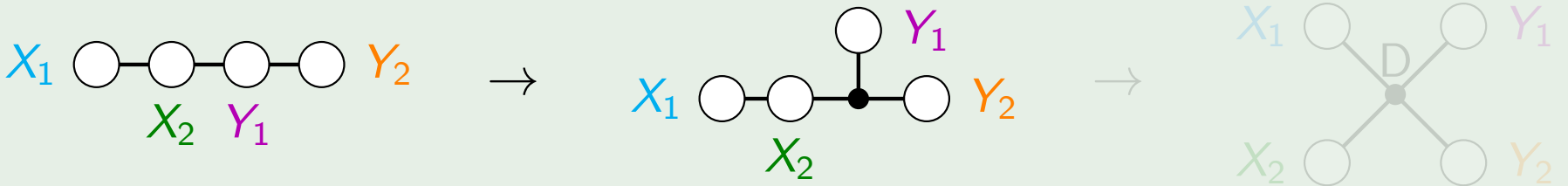

The 3-tree algorithm: an example

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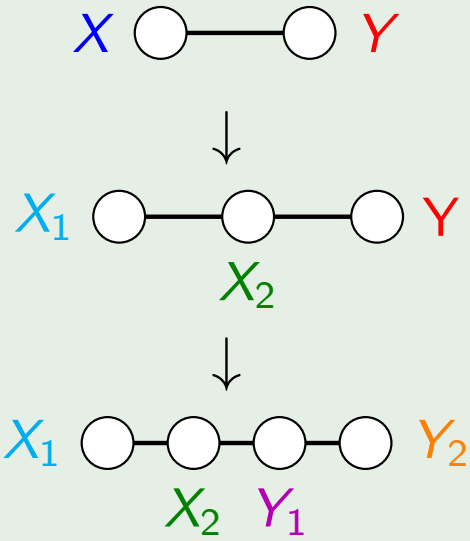

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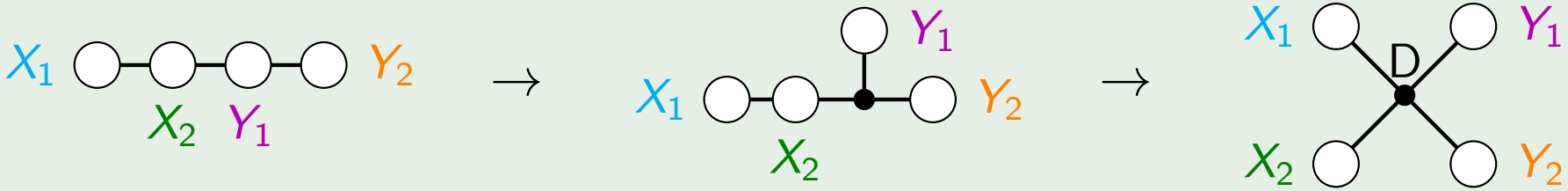

4-trees

Theorem (Aikin and Oxley, 2012)

For every 4-connected matroid with $|E(M)| \geq 17$, there is a tree that displays, up to a natural equivalence, all the non-sequential 4-separations.

- Called a 4-tree
- For 4-separations, it is natural to allow up to two elements to be moved across the 4-separation
- Define "full closure" operator (and non-sequential 4-separations) accordingly
- Whereas a tight 3-flower has no unnecessary petals, for a 4-flower to have no unnecessary petals it also needs to be irredundant

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k-trees

Theorem (Clark and Whittle, 2013)

For any tangle of order k in a connectivity system satisfying a "robustness" condition, there is a tree that displays the non-trivial k-separations with respect to the tangle.

Corollary

For any k-connected matroid M with $|E(M)| \geq 8k - 15$, there is a tree that displays, up to a natural equivalence, all the non-sequential k-separations.

- A k-tree

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- A k-tree

k-equivalence

Corollary (Clark and Whittle, 2013)

For any k-connected matroid M, there is a tree that displays, up to k-equivalence, all the k-separations that are not k-sequential.

Let $(X, E - X)$ be a k-separation.

- $(Y_i)_{i=1}^m$ is a partial k-sequence for X if $|Y_j| \leq k-2$ and $X \cup Y_1 \cup \cdots \cup Y_j$ is k-separating for each $j \in \{1, 2, \ldots, m\}$.
- When $(Y_i)_{i=1}^m$ is maximal, $\mathrm{fcl}_k(X) = X \cup Y_1 \cup \cdots \cup Y_i$
- (X, Y) is k-sequential if $\mathrm{fcl}_k(X) = E(M)$ or $\mathrm{fcl}_k(Y) = E(M)$
- (X, Y) is k-equivalent to (A, B) if $\{\mathrm{fcl}_k(X), \mathrm{fcl}_k(Y)\} = \{\mathrm{fcl}_k(A), \mathrm{fcl}_k(B)\}$

Constructing k-trees

Can we construct a k-tree in polynomial time?

Theorem (B, Semple, 2015)

A k-tree for a k-connected matroid can be constructed in polynomial time.

- First, need to be able find non-sequential k-separations in polynomial time

Lemma (B, Semple, 2015)

A k-separation (U, V) in a k-connected matroid is not k-sequential iff there are k-element sets $U' \subseteq U$ and $V' \subseteq V$ such that neither U' nor V' is contained in a maximal k-sequential set.

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The k-Tree algorithm

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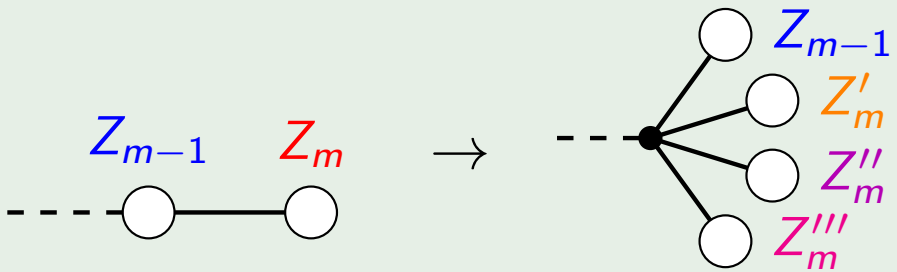
The k-Tree algorithm:

- Similar process to 3-Tree
  1. Find a non-sequential k-separation
  2. ForwardSweep
  3. BackwardSweep
  4. Repeat while there is an unmarked bag
- ForwardSweep process also very similar (replace 3's with k's)
- BackwardSweep is rather more complicated

BackwardSweep complication 1: k-sequential petals

- An end part of a 3-path can break into at most 2 sequential petals
- An end part of a 4-path can break into at most 3 sequential petals

Example



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Lemma (B, Semple, 2015)

No union of three consecutive tight petals in a k-flower is k-sequential.

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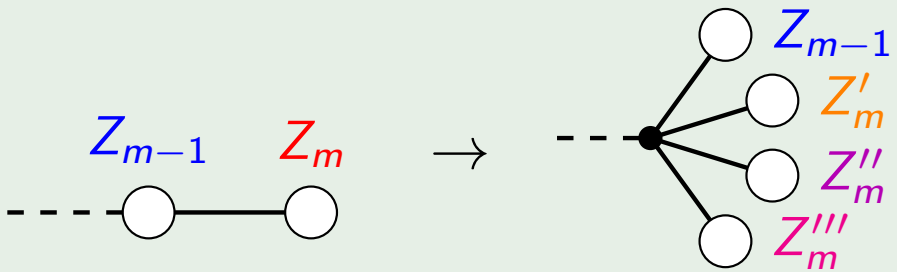

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BackwardSweep complication 2: redundant flowers

- Ensuring the algorithm produces irredundant flowers
  - A k-anemone $\Phi = (X_1, X_2, \ldots, X_n)$ is irredundant if, for all distinct $i, j \in \{1, 2, \ldots, n\}$, there is a non-sequential k-separation (Y, Z) displayed by Φ for which $X_i \subseteq Y$ and $X_j \subseteq Z$
  - Similar for a k-daisy, but with consecutive i, j.
  - A tight 3-flower is irredundant, but a tight k-flower (for $k \geq 4$) may not be
  - A tight irredundant k-flower has no unnecessary petals
- Extra checks, that a certain k-separation exists, are required when refining end petals

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Thanks for your attention