Denser Triangle-Free Binary Matroids

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- If $\delta(G) > \frac{1}{3}n$, then $\chi(G) \leq 4$ *(Brandt-Thomassé, 2006)*.
- For each $\epsilon > 0$, there exists a Δ-free graph G with $\delta(G) > (\frac{1}{3} - \epsilon)|V(G)|$ that has $\chi(G)$ arbitrarily large *(Hajnal, 1973)*.

Bose-Burton, 1966

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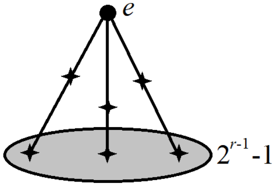

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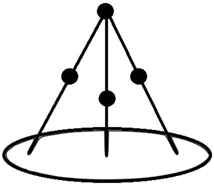

$$r\underbrace{\begin{bmatrix} 1 \\ \hline \\ \\ \end{bmatrix}}_{\frac{1}{2}2^r} \cong AG(r-1,2)$$

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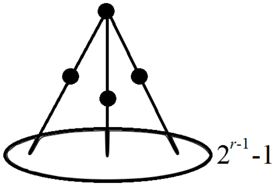

$$r\left[\begin{array}{c} 1 \\ \hline \\ \\ \end{array}\right] \cong AG(r-1,2)$$

$$\tfrac{1}{2}2^r$$

Remark

If a binary matroid M is affine if and only if does not contain any odd cycles.

Goevaerts-Storme, 2006

If M is a simple Δ-free rank-r binary matroid with $|M| > \frac{5}{16}2^r$, then M is affine.

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If M is a simple Δ-free rank-r binary matroid with $|M| > \frac{5}{16}2^r$, then M is affine.

This is also tight:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cong C_5$$

Critical Number

Let M be a simple rank-r binary matroid, considered as a restriction of $PG(r-1,2)$.

We say that M has *critical number* k when the subspaces of $PG(r-1,2)$ contained in in $PG(r-1,2) \setminus M$ have minimum codimension k.

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Remark

If G is a graph, then $M(G)$ has critical number k if and only if $2^{k-1} < \chi(G) \leq 2^k$.

Geelen-Nelson, 2014

For any $\epsilon > 0$, each simple Δ-free binary matroid M with $|M| \geq (\frac{1}{4} + \epsilon)2^{r(M)}$ has bounded critical number.

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Conjecture

Each simple Δ-free binary matroid M with $|M| > \frac{1}{4}2^{r(M)}$ has critical number at most 2.

Geelen-Nelson, 2014

For any $\epsilon > 0$*, each simple* Δ*-free binary matroid M with* $|M| \geq (\frac{1}{4} + \epsilon)2^{r(M)}$ *has bounded critical number.*

Conjecture

Each simple Δ-free binary matroid M with $|M| > \frac{1}{4}2^{r(M)}$ has critical number at most 2.

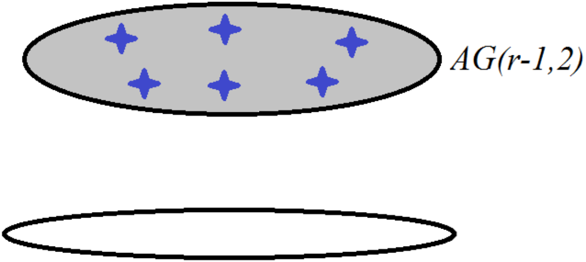
Geelen-Nelson, 2014

For any $\epsilon > 0$*, there exist a simple* Δ*-free binary matroid M with* $|M| \geq (\frac{1}{4} - \epsilon)2^{r(M)}$ *that has arbitrarily large critical number.*

Question

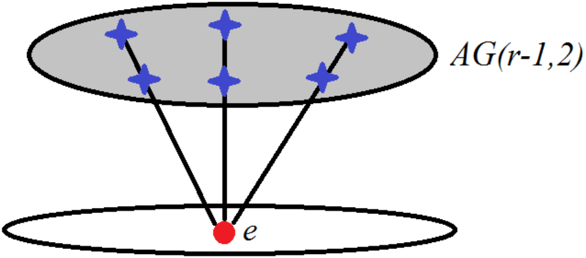
What is the structure of a simple Δ-free rank-r binary matroid M with $|M| > \frac{1}{4}2^r$?

Affine Plus a Point



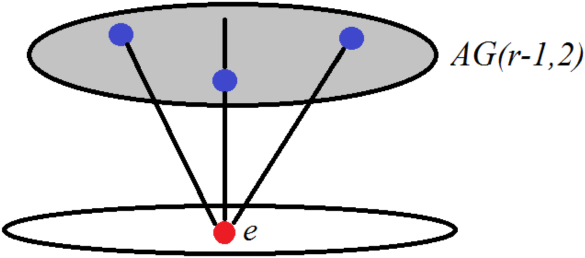
Size: $\frac{1}{2}2^r$

Affine Plus a Point



Size: $\frac{1}{2}2^r + 1$

Affine Plus a Point



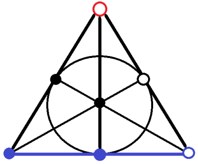
Size: $\frac{1}{4}2^r+1$

Doubling

Let M be a rank r binary matroid given by a matrix A, and with density $\alpha = \frac{|M|}{2^r}$ and critical number c.

$\begin{bmatrix} A \end{bmatrix}$

Rank : r
Size : $|M|$
Density : α
Critical Number : k



$\left[\begin{array}{c|c} 0 & 1 \\ \hline A & A \end{array}\right]$

Rank : $r+1$
Size : $2|M|$
Density : α
Critical Number : k

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$$\left[\ A\ \right]$$

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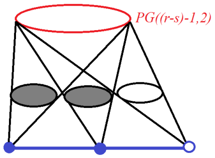

$$\left[\begin{array}{c}\mathbb{Z}_2^s \\ \hline A\end{array}\right]$$

Rank : $r + s$
Size : $2^s|M|$
Density : α
Critical Number : k

Example: Doubling a point

$$AG(r-1,2) \cong \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & \ldots & 1 \\ 0 & 1 & 0 & \ldots & 1 \\ \hline 1 & 1 & 1 & \ldots & 1 \end{array}\right] \cong \left[\frac{\mathbb{Z}_2^{r-1}}{1}\right]$$

is the repeated doubling of a point: $U_{1,1} \cong [1]$

This has fixed density $\frac{1}{2}$

Example: Doubling C_5

$$\left[\begin{array}{c|c|c|c|c} 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & \dots & 1\,1\,1\,1\,1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 1\,1\,1\,1\,1 & \dots & 1\,1\,1\,1\,1 \\ 0\,0\,0\,0\,0 & 1\,1\,1\,1\,1 & 0\,0\,0\,0\,0 & \dots & 1\,1\,1\,1\,1 \\ \hline 1\,0\,0\,0\,1 & 1\,0\,0\,0\,1 & 1\,0\,0\,0\,1 & \dots & 1\,0\,0\,0\,1 \\ 0\,1\,0\,0\,1 & 0\,1\,0\,0\,1 & 0\,1\,0\,0\,1 & \dots & 0\,1\,0\,0\,1 \\ 0\,0\,1\,0\,1 & 0\,0\,1\,0\,1 & 0\,0\,1\,0\,1 & \dots & 0\,0\,1\,0\,1 \\ 0\,0\,0\,1\,1 & 0\,0\,0\,1\,1 & 0\,0\,0\,1\,1 & \dots & 0\,0\,0\,1\,1 \end{array}\right] \cong \left[\begin{array}{c} \mathbb{Z}_2^{r-4} \\ \hline 1\,0\,0\,0\,1 \\ 0\,1\,0\,0\,1 \\ 0\,0\,1\,0\,1 \\ 0\,0\,0\,1\,1 \end{array}\right]$$

is the repeated doubling of a five cycle: $C_5 \cong \begin{bmatrix} 1\,0\,0\,0\,1 \\ 0\,1\,0\,0\,1 \\ 0\,0\,1\,0\,1 \\ 0\,0\,0\,1\,1 \end{bmatrix}$

Note $|C_5| = \frac{1}{4}2^4 + 1 = \frac{5}{16}2^4$. So this has fixed density $\frac{5}{16}$.

Homomorphism

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Lemma

If N is a simple Δ-free rank-s binary matroid and M admits a homomorphism to N, then M is also Δ-free.

Main Theorem

If M be a simple Δ-free rank-r binary matroid with $M > \frac{33}{128}2^r$, then M has a homomorphism to a simple Δ-free binary matroid N of rank at most 6.

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$$\mathbf{r = 1}, \alpha = \mathbf{\tfrac{1}{2}} : [1]$$

$$\mathbf{r = 4}, \alpha = \mathbf{\tfrac{5}{16}} : \begin{bmatrix} 10001 \\ 01001 \\ 00101 \\ 00011 \end{bmatrix}$$

$$\mathbf{r = 5}, \alpha = \mathbf{\tfrac{9}{32}} : \begin{bmatrix} 000001111 \\ 100011001 \\ 010010101 \\ 001010011 \\ 000110000 \end{bmatrix}$$

$$\mathbf{r = 6}, \alpha = \mathbf{\tfrac{17}{64}} : \begin{bmatrix} 00000000011111111 \\ 00000111100001111 \\ 10001100010000111 \\ 01001010001001011 \\ 00101001000101101 \\ 00011000100011110 \end{bmatrix}, \begin{bmatrix} 00000000011111111 \\ 00000111100001111 \\ 10001100110011000 \\ 01001010101010100 \\ 00101001100000010 \\ 00011000000110001 \end{bmatrix}, \begin{bmatrix} 00000000011111111 \\ 00000111100001111 \\ 10001100110011001 \\ 01001010101010000 \\ 00101001100000101 \\ 00011000000110011 \end{bmatrix},$$

$$\begin{bmatrix} 00000000011111111 \\ 00000111100001111 \\ 10001100110011001 \\ 01001010101010101 \\ 00101001100110000 \\ 00011000000000011 \end{bmatrix}, \begin{bmatrix} 00000000011111111 \\ 00000111100001111 \\ 10001100110011001 \\ 01001010101010101 \\ 00101001100110011 \\ 00011000000000000 \end{bmatrix}$$

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Corollaries

- If $|M| > \frac{9}{32}2^r$, then M has a homomorphism to a simple Δ-free binary matroid of rank at most 4.
- If $|M| > \frac{17}{64}2^r$, then M has a homomorphism to a simple Δ-free binary matroid of rank at most 5.

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- If $|M| > \frac{17}{64}2^r$, then M has a homomorphism to a simple Δ-free binary matroid of rank at most 5.
- If $|M| > \frac{33}{128}2^r$, then M has critical number at most 2.

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This is tight:

$$\left[\begin{array}{ccccc|ccccc|ccc|cccc|ccc|ccccc|ccc|ccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&0&0&0&1&1&1&1&0&0&0&1&1&1&1&1&0&0&0&1&1&1&1&1\\
\hline
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0&0&0&1&1&0&0&0&1&1&0&0&1&0&0&1&1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1&1
\end{array}\right]$$

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If M be a simple Δ-free rank-r binary matroid with $M > \frac{33}{128}2^r$, then M has a homomorphism to a simple Δ-free binary matroid N of rank at most 6.

Proof Sketch of Main Theorem

We may assume that:

- $|M| > \frac{33}{128}2^r$,
- $r \geq 7$,
- M is maximal Δ-free, and
- M does not admit a homomorphism to a Δ-free matroid in lower rank.

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Thus $|M| > \frac{1}{4}2^r + 1$.

Proof Sketch of Main Theorem (Cont.)

Geelen, 2014

If M is a simple Δ-free rank-r binary matroid with $|M| > \frac{7}{64}2^r$, then M is either affine or contains C_5.

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If M is a simple Δ-free rank-r binary matroid with $|M| > \frac{7}{64}2^r$, then M is either affine or contains C_5.

If M is affine, we have a homomorphism to a point. Thus we may assume M contains C_5.

Proof Sketch of Main Theorem (Cont.)

- M contains C_5
- $|M| > \frac{1}{4}2^r + 1$

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Claim: M has a 10-element rank-5 restriction containing C_5.

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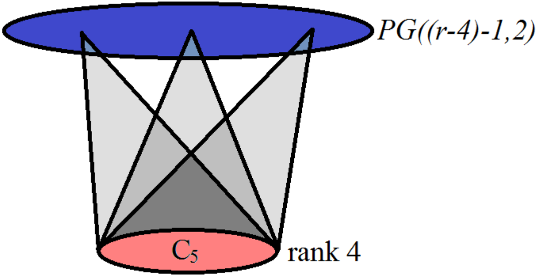

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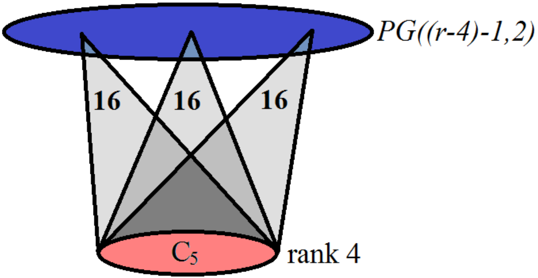

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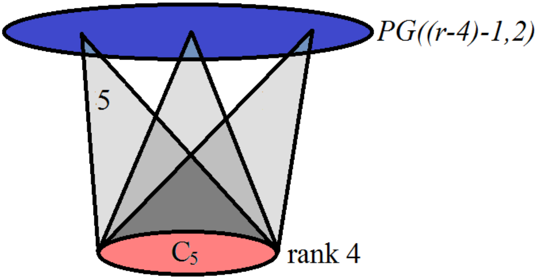

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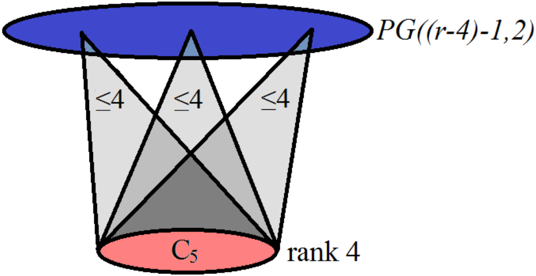

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Claim: The only Δ-free, 10-element matroid containing C_5 and with rank-5 is the doubling of C_5.

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$$\left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 1 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & 1 & & & & & \end{array}\right]$$

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This corresponds to a 5 element stable set in Clebsch Graph.

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$$\left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

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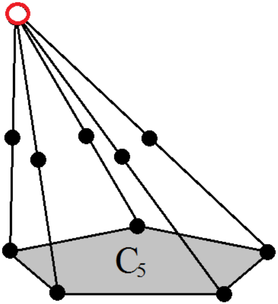
Proof Sketch of Main Theorem (Cont.)

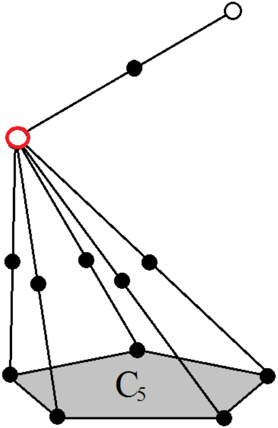
M contains:



rank 5

Proof Sketch of Main Theorem (Cont.)

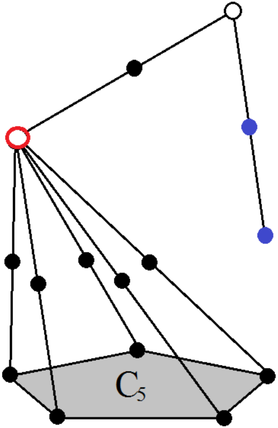
M contains:



rank 6

Proof Sketch of Main Theorem (Cont.)

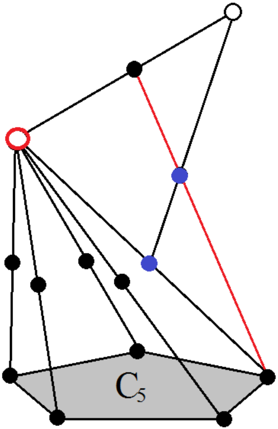
M contains:



rank 6 or 7

Proof Sketch of Main Theorem (Cont.)

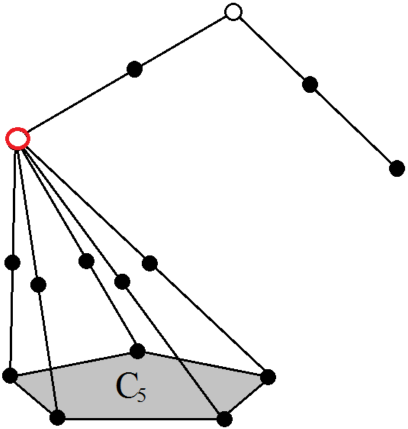
M contains:



rank 6

Proof Sketch of Main Theorem (Cont.)

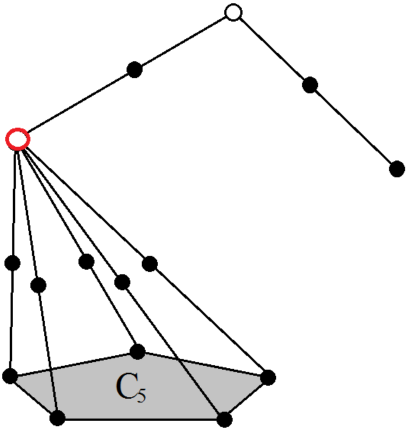
M contains:



rank 7

Proof Sketch of Main Theorem (Cont.)

M contains:



rank 7

Call this Matroid *Q*.

Proof Sketch of Main Theorem (Cont.)

- M contains Q
- $|M| > \frac{33}{128}2^r$

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Claim: M cannot have rank $r = 7$.

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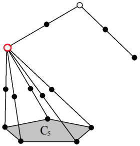
Best we can do is density exactly $\frac{33}{128}$:

$$\left[\begin{array}{ccccc|ccccc|ccc|cccc|ccc|ccccc|ccc|ccccc}
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&0&0&0&0&0&0&\mathbf{0}&1&1&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{1}&1&1&1&1\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&1&1&1&1&1&1&\mathbf{1}&0&0&0&0&0&0&0&\mathbf{0}&1&1&1&\mathbf{1}&1&1&1&1\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0&0&1&1&1&\mathbf{1}&0&0&0&1&1&1&1&\mathbf{1}&0&0&0&\mathbf{1}&1&1&1&1\\
\hline
\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&0&0&0&0&0&0&\mathbf{0}&0&0&0&1&0&0&0&\mathbf{1}&0&0&0&\mathbf{1}&0&0&0&1\\
\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&1&0&0&1&0&0&\mathbf{1}&1&0&0&0&1&0&0&\mathbf{1}&1&0&0&\mathbf{0}&1&0&0&1\\
\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&0&1&0&0&1&0&\mathbf{1}&0&1&0&0&0&1&0&\mathbf{1}&0&1&0&\mathbf{0}&0&1&0&1\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&0&0&1&0&0&1&\mathbf{1}&0&0&1&0&0&0&1&\mathbf{1}&0&0&1&\mathbf{0}&0&0&1&1
\end{array}\right]$$

Q :



Proof Sketch of Main Theorem (Cont.)

- M contains Q
- $|M| > \frac{33}{128} 2^r$

Claim: M cannot have rank $r = 7$.

Best we can do is density exactly $\frac{33}{128}$:

$$\left[\begin{array}{ccccc|ccccc|ccc|cccc|ccc|ccccc|ccc|ccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&0&0&0&1&1&1&1&0&0&0&1&1&1&1&1&0&0&0&1&1&1&1&1\\
\hline
1&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
0&1&0&0&1&0&1&0&0&1&1&0&0&1&0&0&1&1&0&0&0&1&0&0&1&1&0&0&0&1&0&0&1\\
0&0&1&0&1&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0&0&0&1&0&1\\
0&0&0&1&1&0&0&0&1&1&0&0&1&0&0&1&1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1&1
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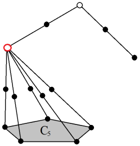

So $r \geq 8$, and $|cl_M(Q)| \leq 33$.

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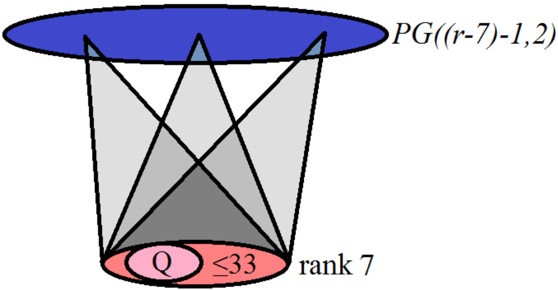

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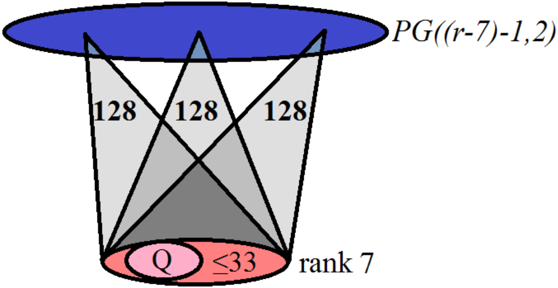

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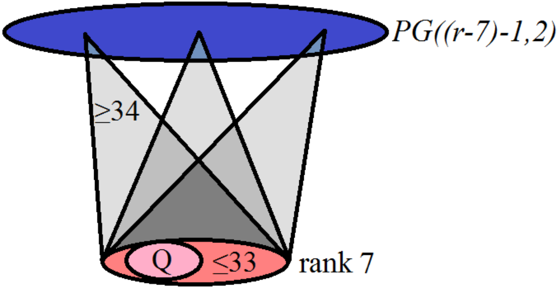

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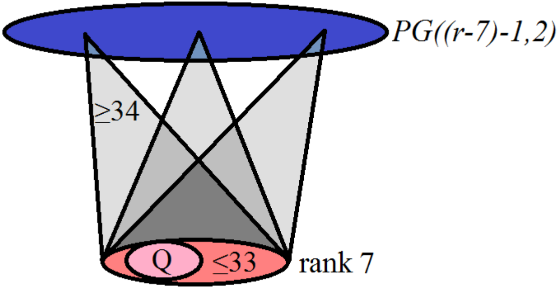

Claim: There is no simple Δ-free rank-8 binary matroid containing Q and at least 34 elements in the complement of the closure of Q.

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Conjecture

- For any $\epsilon > 0$, there is a finite set S_ϵ of simple Δ-free binary matroids, such that if $|M| > (\frac{1}{4} + \epsilon)2^r$, then M has a homomorphism to some $N \in S_\epsilon$.

Thank You