Intertwining connectivity in matroids

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Connectivity function

Let (X, Y) be a partition of the ground set of a matroid M. Define

$$\lambda_M(X, Y) := r_M(X) + r_M(Y) - r(M).$$

For disjoint subsets Q, R of $E(M)$, the *connectivity between Q and R* is

$$\kappa_M(Q, R) := \min\{\lambda_M(Q', R') : Q \subseteq Q',\ R \subseteq R'\}.$$

Connectivity of Matroids

For every minor N of M with $Q \cup R \subseteq E(N)$, we have

$$\kappa_N(Q, R) \leq \kappa_M(Q, R).$$

Tutte's Linking Theorem

For every $e \in E(M) - (Q \cup R)$, either

$$\kappa_{M \backslash e}(Q, R) = \kappa_M(Q, R)$$

or

$$\kappa_{M/e}(Q, R) = \kappa_M(Q, R).$$

Let $Q, R, S, T \subseteq E(M)$ with $Q \cap R = S \cap T = \emptyset$. Set

$$k := \kappa_M(Q, R), \quad \ell := \kappa_M(S, T),$$

$$F := E(M) - (Q \cup R \cup S \cup T).$$

Conjecture (Jim Geelen)

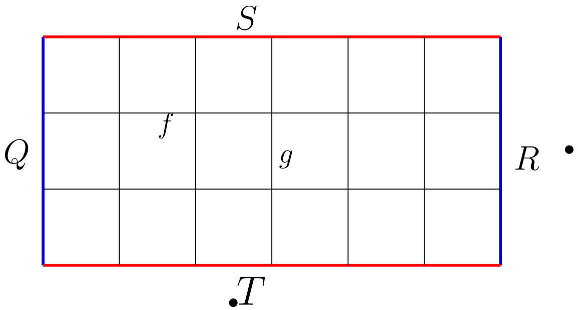
If F is sufficiently large, then there is an element e of M such that

(i) $\kappa_{M\backslash e}(Q, R) = k$ *and* $\kappa_{M\backslash e}(S, T) = \ell$; *or*

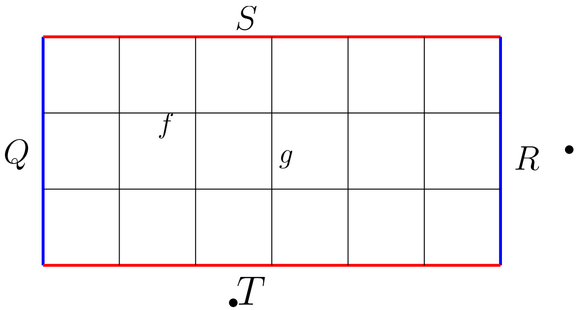
(ii) $\kappa_{M/e}(Q, R) = k$ *and* $\kappa_{M/e}(S, T) = \ell$.

Why should F be sufficiently large?

An example



An example



No element can be removed to keep the both connectivities.

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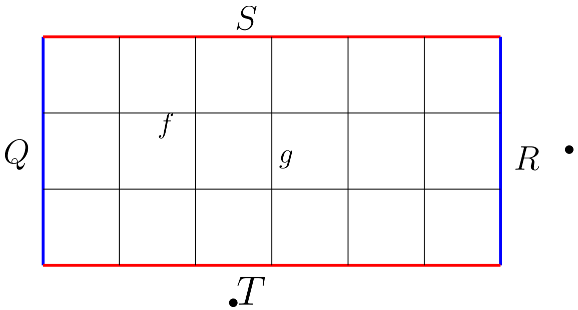

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So $|F| \geq 2k\ell - \ell - k + 1$.

Conjecture

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We proved

Theorem

When $|F| \geq (2\ell + 1)2^{2k+1}$, *Geelen's Conjecture holds.*

Proof sketch

First we proved

Lemma

When $|S| = |T| = \ell$ *and* $|F| \geq (2\ell + 1)2^{2k+1}$*, Geelen's conjecture holds.*

Proof sketch

A partition (Q', R') of $E(M)$ is $Q - R$*-separating of order* $k + 1$ if $Q \subseteq Q'$, $R \subseteq R'$ and $\lambda_M(A) \leq k$.

$\kappa_{M\backslash e}(Q, R) = \kappa_M(Q, R) \iff e$ is *deletable with respect to* (Q, R).

$\kappa_{M/e}(Q, R) = \kappa_M(Q, R) \iff e$ is *contractible with respect to* (Q, R).

e is both deletable and contractible with respect to (Q, R) $\iff e$ is *flexible with respect to* (Q, R).

Proof sketch

Theorem (Stefan van Zwam and Tony Huynh, 2013)

Let F consist of non-flexible elements with respect to (Q, R). Then there are an ordering $(f_1, \ldots, f_n)$ of F and a sequence $(A_1, \ldots, A_n)$ of subsets of $E(M)$ such that $\forall i \in \{1, \ldots, n\}$

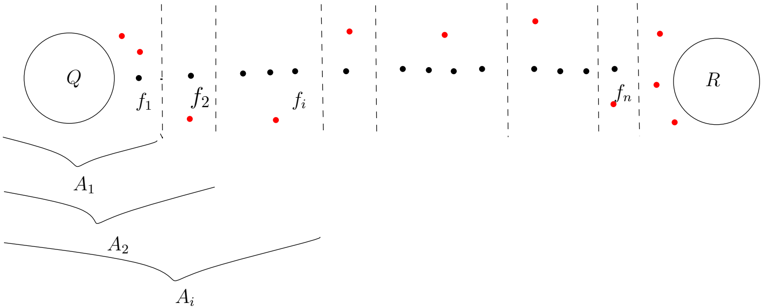
(i) A_i *is* $Q - R$*-separating of order* $k + 1$*;*

(ii) $A_i \subseteq A_{i+1}$*;*

(iii) $A_i \cap F = \{f_1, \ldots, f_i\}$*;*

(iv) $f_i \in \mathrm{cl}(A_i - \{f_i\}) \cap \mathrm{cl}(E(M) - A_i)$ *or* $f_i \in \mathrm{cl}^*(A_i - \{f_i\}) \cap \mathrm{cl}^*(E(M) - A_i)$.

Proof sketch



$$\lambda_M(A_i) = \kappa_M(Q, R), \quad \forall 1 \le i \le n.$$

When f_i is non-deletable, $f_i \in \mathrm{cl}^*(A_i - \{f_i\}) \cap \mathrm{cl}^*(E(M) - A_i)$;

When f_i is non-contractible, $f_i \in \mathrm{cl}(A_i - \{f_i\}) \cap \mathrm{cl}(E(M) - A_i)$.

Proof sketch

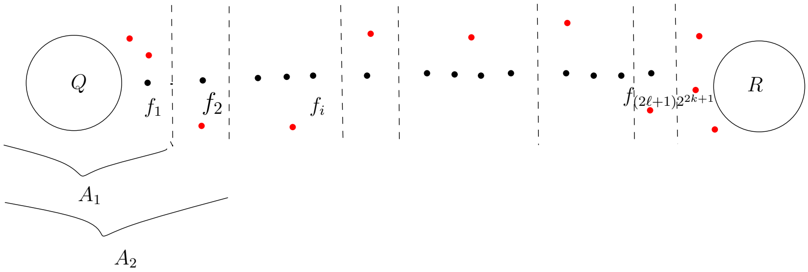
Note that

(1) Each element in F is non-flexible with respect to (Q, R) and non-flexible with respect to (S, T).

(2) $e \in F$ is deletable (or contractible) with respect to (Q, R) if and only if e is contractible (or deletable) with respect to (S, T).

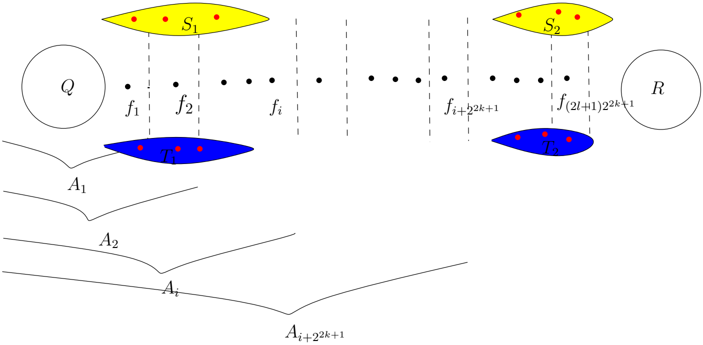
Proof sketch

Let $(A_1, \ldots, A_{(2\ell+1)2^{2k+1}})$ be the nested sequence of $Q - R$ separating sets.



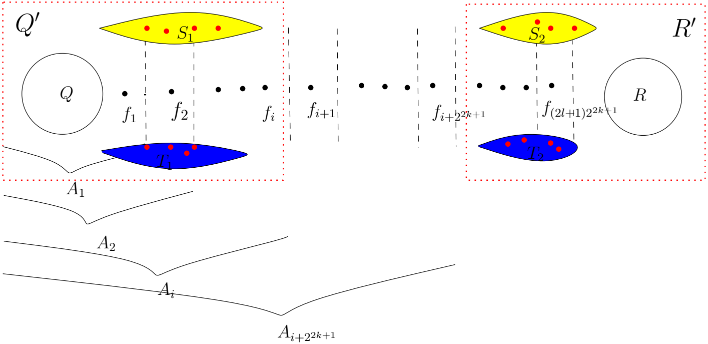
Proof sketch

Since $|S| = |T| = \ell$,



$$S = S_1 \cup S_2, \quad T = T_1 \cup T_2.$$

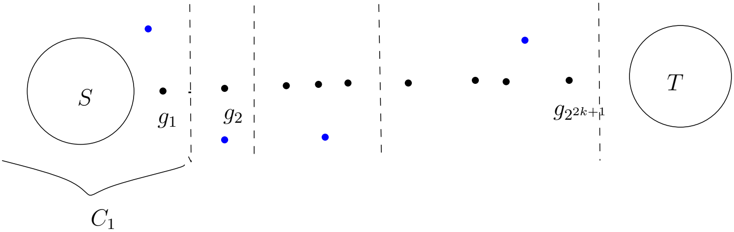
Proof sketch



$$f_j' := f_{i+j}, \text{ for any } 1 \le j \le 2^{2k+1},$$
$$F' := \{f_1', \dots, f_{2^{2k+1}}'\},$$

(Q', F', R') is a partition of $E(M)$.

Proof sketch



By duality assume g_1 is a deletable element with respect to (S, T).

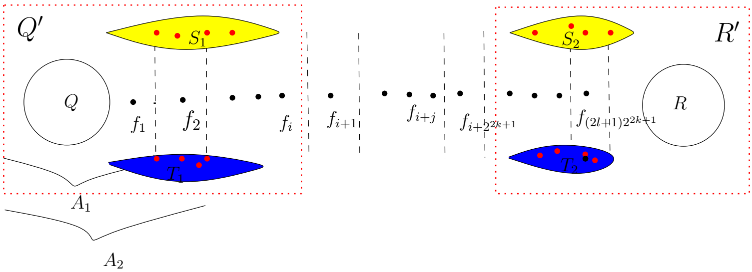
$$(a)\ g_1 \in \mathrm{cl}(C_1 - \{g_1\}) \quad + \quad C_1 - \{g_1\} \subseteq Q' \cup R'$$
$$\implies g_1 \in \mathrm{cl}(Q' \cup R'),$$

(b) g_1 is contractible with respect to (Q, R)

$$\implies g_1 \notin \mathrm{cl}(Q') \cup \mathrm{cl}(R').$$

$$(a) \quad + \quad (b) \implies \sqcap_M(Q' \cup \{g_1\}, R') = \sqcap_M(Q', R') + 1.$$

Proof sketch



Assume $g_1 = f_j^{'} = f_{i+j}$.

If $j \leq 2^{2k}$ then set $Q^{''} := A_{i+j}, R^{''} := R^{'}$.

Else if $j > 2^{2k}$ then set $Q^{''} := Q^{'}, R^{''} := A^c_{i+j-1}$.

No matter which case happens, set $F^{''} := E(M) - (Q^{''} \cup R^{''})$.

Evidently, $|F^{''}| \geq 2^{2k}$ as $|F^{'}| = 2^{2k+1}$.

Proof sketch

Replacing $Q^{'}, R^{'}, F^{'}$ with $Q^{''}, R^{''}, F^{''}$ respectively and repeating the above analysis $2k$ times, there are numbers j_1, j_2 with $2k+1 \leq j_1 \leq j_2 \leq 2^{2k+1}$ such that $\sqcap_M(A_{i+j_1}, A^c_{i+j_2}) \geq k+1$ or $\sqcap_{M^*}(A_{i+j_1}, A^c_{i+j_2}) \geq k+1$, a contradiction to the fact that $\lambda(A_{i+j_1}) = k$. That is, we prove

Lemma

When $|S| = |T| = \ell$ and $|F| \geq (2\ell+1)2^{2k+1}$, Geelen's conjecture holds.

Proof sketch

Lemma (Geelen, Gerards, Whittle, 2007.)

Let S, T be disjoint subsets of $E(M)$. There exist sets $S_1 \subseteq S, T_1 \subseteq T$ with $|S_1| = |T_1| = \kappa_M(S, T)$.

$\implies \kappa_M(S_1, T_1) = |S_1| = |T_1| = \ell$.

$\exists e_1 \in E(M) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_1 \in \{M \backslash e_1, M/e_1\}$ we have $\kappa_{M_1}(Q, R) = k$ and $\kappa_{M_1}(S_1, T_1) = \ell$.

If $e_1 \in F$, Geelen's conjecture holds.

Proof sketch

If $e_1 \notin F$, since $F \subseteq E(M_1) - (Q \cup R \cup S_1 \cup T_1)$, $\exists e_2 \in E(M_1) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_2 \in \{M_1 \backslash e_2, M_1/e_2\}$ we have $\kappa_{M_2}(Q, R) = k$ and $\kappa_{M_2}(S_1, T_1) = \ell$.

Assume $M_2 = M_1 \backslash e_2$. Then $\kappa_{M \backslash e_2}(Q, R) = k$ and $\kappa_{M \backslash e_2}(S_1, T_1) = \ell$ as $\kappa_M(Q, R) = k$ and $\kappa_M(S_1, T_1) = \ell$.

Thus, when $e_2 \in F$, the lemma holds. So we may assume that $e_2 \notin F$.

Since $(S \cup T) - (S_1 \cup T_1)$ is finite, repeating the above analysis several times we can always find an element $e \in F$ to be removed to keep the connectivities.

Thank You!