Inductive tools for delta-matroids

Carolyn Chun[1] Deborah Chun[2] Steven Noble[1]

[1]Brunel University London

[2]West Virginia University Institute of Technology

July 22, 2014

Matroids*

$M = (E, \mathcal{B})$

(Symmetric Exchange Axiom) $B_1, B_2 \in \mathcal{B}$ and
$e \in B_1 \triangle B_2 \Rightarrow f \in B_1 \triangle B_2$ where $B_1 \triangle \{e, f\} \in \mathcal{B}$

1) $\mathcal{B} \neq \emptyset$
2) (SEA) holds for $\mathcal{B}$

Matroids*

$M = (E, \mathcal{B})$

(Symmetric Exchange Axiom) $B_1, B_2 \in \mathcal{B}$ and
$e \in B_1 \triangle B_2 \Rightarrow f \in B_1 \triangle B_2$ where $B_1 \triangle \{e, f\} \in \mathcal{B}$

1) $\mathcal{B} \neq \emptyset$
2) (SEA) holds for $\mathcal{B}$
3) $B_1, B_2 \in \mathcal{B} \Rightarrow |B_1| = |B_2|$

Matroids*

$M = (E, \mathcal{B})$

(Symmetric Exchange Axiom) $B_1, B_2 \in \mathcal{B}$ and
$e \in B_1 \triangle B_2 \Rightarrow f \in B_1 \triangle B_2$ where $B_1 \triangle \{e, f\} \in \mathcal{B}$

1) $\mathcal{B} \neq \emptyset$
2) (SEA) holds for $\mathcal{B}$
3) $B_1, B_2 \in \mathcal{B} \Rightarrow |B_1| = |B_2|$

(* Defined by Hassler Whitney in 1935.)

delta-matroids*

$D = (E, \mathcal{F})$

(Symmetric Exchange Axiom) $F_1, F_2 \in \mathcal{F}$ and
$e \in F_1 \triangle F_2 \Rightarrow f \in F_1 \triangle F_2$ where $F_1 \triangle \{e, f\} \in \mathcal{F}$

1) $\mathcal{F} \neq \emptyset$
2) (SEA) holds for $\mathcal{F}$

delta-matroids*

$D = (E, \mathcal{F})$

(Symmetric Exchange Axiom) $F_1, F_2 \in \mathcal{F}$ and
$e \in F_1 \triangle F_2 \Rightarrow f \in F_1 \triangle F_2$ where $F_1 \triangle \{e, f\} \in \mathcal{F}$

1) $\mathcal{F} \neq \emptyset$
2) (SEA) holds for $\mathcal{F}$

(* Defined by André Bouchet in 1986. Also known as *pseudomatroids*, defined by Chandrasekaran and Kabadi in 1988.)

Minors

$$D = (E, \mathcal{F}), e \in E$$

$$D \backslash e = (E - e, \{F | F \in \mathcal{F} \text{ and } e \notin F\})$$

$$D / e = (E - e, \{F - e | F \in \mathcal{F} \text{ and } e \in F\})$$

Minors

$$D = (E, \mathcal{F}), e \in E$$

$$D \backslash e = (E - e, \{F | F \in \mathcal{F} \text{ and } e \notin F\}) \; e \text{ not a coloop}$$

$$D / e = (E - e, \{F - e | F \in \mathcal{F} \text{ and } e \in F\}) \; e \text{ not a loop}$$

Minors

$D \backslash e = D/e$ if e is a loop or coloop

Connectivity

For $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$:

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{B_1 \cup B_2 | B_i \in \mathcal{B}_i\})$$

Connectivity

For $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$:

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{B_1 \cup B_2 | B_i \in \mathcal{B}_i\})$$

TFAE:

1. M is connected.

Connectivity

For $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$:

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{B_1 \cup B_2 | B_i \in \mathcal{B}_i\})$$

TFAE:

1. M is connected.
2. $M \neq M_1 \oplus M_2$.

Connectivity

For $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$:

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{B_1 \cup B_2 | B_i \in \mathcal{B}_i\})$$

TFAE:

1. M is connected.
2. $M \neq M_1 \oplus M_2$.
3. $e, f \in E(M) \Rightarrow \{e, f\}$ is contained in a circuit.

Connectivity

For $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$:

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{B_1 \cup B_2 | B_i \in \mathcal{B}_i\})$$

TFAE:

1. M is connected.
2. $M \neq M_1 \oplus M_2$.
3. $e, f \in E(M) \Rightarrow \exists B, B \triangle \{e, f\} \in \mathcal{B}(M)$.

Connectivity

For $D_1 = (E_1, \mathcal{F}_1)$ and $D_2 = (E_2, \mathcal{F}_2)$:

$$D_1 \oplus D_2 = (E_1 \cup E_2, \{F_1 \cup F_2 | F_i \in \mathcal{F}_i\})$$

Connectivity

For $D_1 = (E_1, \mathcal{F}_1)$ and $D_2 = (E_2, \mathcal{F}_2)$:

$$D_1 \oplus D_2 = (E_1 \cup E_2, \{F_1 \cup F_2 | F_i \in \mathcal{F}_i\})$$

TFAE:

1. D is connected.

Connectivity

For $D_1 = (E_1, \mathcal{F}_1)$ and $D_2 = (E_2, \mathcal{F}_2)$:

$$D_1 \oplus D_2 = (E_1 \cup E_2, \{F_1 \cup F_2 | F_i \in \mathcal{F}_i\})$$

TFAE:

1. D is connected.
2. $D \neq D_1 \oplus D_2$.

Connectivity

For $D_1 = (E_1, \mathcal{F}_1)$ and $D_2 = (E_2, \mathcal{F}_2)$:

$$D_1 \oplus D_2 = (E_1 \cup E_2, \{F_1 \cup F_2 | F_i \in \mathcal{F}_i\})$$

TFAE:

1. D is connected.
2. $D \neq D_1 \oplus D_2$.
3. $e, f \in E(D) \Rightarrow \exists F, F \triangle \{e \triangle f\} \in \mathcal{F}(D)$.

Connectivity

For $D_1 = (E_1, \mathcal{F}_1)$ and $D_2 = (E_2, \mathcal{F}_2)$:

$$D_1 \oplus D_2 = (E_1 \cup E_2, \{F_1 \cup F_2 | F_i \in \mathcal{F}_i\})$$

TFAE:

1. D is connected.
2. $D \neq D_1 \oplus D_2$.
3. $e \sim f$.

Even delta-matroids

TFAE:

1. D is even.

Even delta-matroids

TFAE:

1. D is even.
2. The feasible sets of D have the same parity.

Even delta-matroids

TFAE:

1. D is even.
2. The feasible sets of D have the same parity.
3. $(\{a\}, \{\emptyset, \{a\}\}) \not\preceq D$.

Even delta-matroids

TFAE:

1. D is even.
2. The feasible sets of D have the same parity.
3. $(\{a\}, \{\emptyset, \{a\}\}) \not\preceq D$.

Lemma

$\sim$ *is an equivalence relation for even delta-matroids.*

Even delta-matroids

$C(e)$ is the equivalence class of e

Even delta-matroids

$C(e)$ is the equivalence class of e

Lemma

D is even and $C(e) = X$ and $X \notin \{\emptyset, E(D)\}$ $\iff$
$D = (X, \{F \cap X | F \in \mathcal{F}(D)\}) \oplus (E - X, \{F - X | F \in \mathcal{F}(D))$

$$(C(e_1), \mathcal{F}_1) \oplus (C(e_2), \mathcal{F}_2) \oplus (C(e_3), \mathcal{F}_3) \oplus \dots$$

Even delta-matroids

Theorem (CCN 2014)

D is even and connected and $e \in E(D) \implies D \backslash e$ or D/e is connected.

Proof.

Even delta-matroids

Theorem (CCN 2014)

D is even and connected and $e \in E(D) \implies D\backslash e$ or D/e is connected.

Proof.

1. Show that $\sim$ is an equivalence relation.
2. Show that $\sim$ has one equiv. class $\iff D$ is connected.
3. Show that, if D is connected and $a \nsim b$ in $D\backslash e$, then $a \sim b$ in D/e.

Even delta-matroids

Theorem (CCN 2014)

D is even and connected and $e \in E(D) \implies D \backslash e$ or D/e is connected.

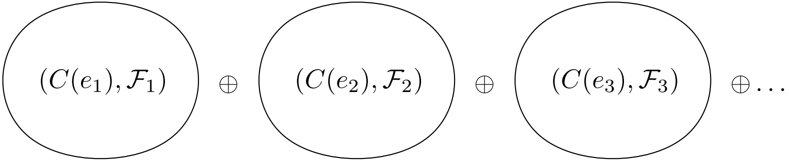
Proof.

1. Show that $\sim$ is an equivalence relation. ✓
2. Show that $\sim$ has one equiv. class $\iff$ D is connected.
3. Show that, if D is connected and $a \nsim b$ in $D \backslash e$, then $a \sim b$ in D/e.

Even delta-matroids

Lemma

D is even and $C(e) = X \iff$
$D = (X, \{F \cap X | F \in \mathcal{F}(D)\}) \oplus (E - X, \{F - X | F \in \mathcal{F}(D))$



Even delta-matroids

Theorem (CCN 2014)

D is even and connected and $e \in E(D) \implies D \backslash e$ or D/e is connected.

Proof.

1. Show that $\sim$ is an equivalence relation. ✓
2. Show that $\sim$ has one equiv. class $\iff$ D is connected. ✓
3. Show that, if D is connected and $a \nsim b$ in $D \backslash e$, then $a \sim b$ in D/e.

Even delta-matroids

Lemma

D is connected and $a \nsim b$ in $D \backslash e \implies a \sim b$ in D/e.

Even delta-matroids

Lemma

D is connected and $a \nsim b$ in $D \backslash e \implies a \sim b$ in D/e.

Proof.

$a \sim b$ in $D \implies F, F \triangle \{a, b\} \in \mathcal{F}(D)$.

Even delta-matroids

Lemma

D is connected and $a \nsim b$ in $D \backslash e \implies a \sim b$ in D/e.

Proof.

$a \sim b$ in $D \implies F, F \triangle \{a, b\} \in \mathcal{F}(D)$.
$a \nsim b$ in $D \backslash e \implies e \in F$.

Even delta-matroids

Lemma

D is connected and $a \nsim b$ in $D \backslash e \implies a \sim b$ in D/e.

Proof.

$a \sim b$ in $D \implies F, F \triangle \{a, b\} \in \mathcal{F}(D)$.
$a \nsim b$ in $D \backslash e \implies e \in F$.
Thus $F - e, (F - e) \triangle \{a, b\} \in \mathcal{F}(D/e)$. □

Even delta-matroids chain theorem

Theorem (CCN 2014)

D is even and connected and $e \in E(D) \implies D \backslash e$ or D/e is connected.

Even delta-matroids chain theorem

Theorem (CCN 2014)

D is even and connected and $e \in E(D) \implies D \backslash e$ or D/e is connected.

Corollary (Tutte 1966)

M is connected and $e \in E(M) \implies M \backslash e$ or M/e is connected.

Odd delta-matroids

$$
\begin{array}{ccc} & \{a,b\} & \\ \{a\} & & \{b\} \\ & \emptyset & \end{array}
=
\begin{array}{c} \{a\} \\ \\ \emptyset \end{array}
\oplus
\begin{array}{c} \{b\} \\ \\ \emptyset \end{array}
$$

Odd delta-matroids

$\{a, b\}$

$\{a\}$ $\{b\}$ $=$ $\{a\}$ / $\emptyset$ $\oplus$ $\{b\}$ / $\emptyset$

$\emptyset$

D is connected $\iff \forall a, b \in E(D)$, there is $F \in \mathcal{F}(D)$ such that

1. $F, F \triangle \{a \triangle b\} \in \mathcal{F}(D)$ and

Odd delta-matroids

$$\begin{array}{ccc} & \{a,b\} & \\ \{a\} & & \{b\} \\ & \emptyset & \end{array} = \begin{array}{c} \{a\} \\ \\ \emptyset \end{array} \oplus \begin{array}{c} \{b\} \\ \\ \emptyset \end{array}$$

D is connected $\iff$ $\forall a, b \in E(D)$, there is $F \in \mathcal{F}(D)$ such that

1. $F, F\triangle\{a\triangle b\} \in \mathcal{F}(D)$ and
2. $F\triangle\{a\}$ or $F\triangle\{b\}$ is not in $\mathcal{F}(D)$

Odd delta-matroids

$$\{a, b, c\} \quad \{a, b, d\} \quad \{a, c, d\}$$

$$\{a, b\} \quad \{a, d\} \quad \{b, c\} \quad \{b, d\} \quad \{c, d\}$$

$$\{b\} \quad \{c\} \quad \{d\}$$

$$\emptyset$$

Odd delta-matroids

$$\{a,b,c\} \quad \{a,b,d\} \quad \{a,c,d\}$$

$$\{a,b\} \quad \{a,d\} \quad \{b,c\} \quad \{b,d\} \quad \{c,d\}$$

$$\{b\} \quad \{c\} \quad \{d\}$$

$$\emptyset$$

D/b and $D\backslash b$ are disconnected.

Even delta-matroids splitter theorem

Theorem (CCN 2014)

D is even and connected with connected minor D', and $e \in E(D) - E(D') \implies D \backslash e$ or D/e is connected with D' as a minor.

Even delta-matroids splitter theorem

Theorem (CCN 2014)

D is even and connected with connected minor D', and $e \in E(D) - E(D') \implies D \backslash e$ or D/e is connected with D' as a minor.

Corollary (Tutte 1966)

M is even and connected with connected minor M', and $e \in E(M) - E(M') \implies M \backslash e$ or M/e is connected with M' as a minor.