Highly connected matroids in minor-closed classes

Jim Geelen[*]

Bert Gerards

Geoff Whittle

[*] University of Waterloo

Seymour's Decomposition Theorem

If M is an internally 4-connected regular matroid with $|M| \geqslant 11$, then M is either graphic or cographic.

Let $\mathbb{F}$ be a finite field and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids.

Question What is the structure of the highly connected matroids in $\mathcal{M}$?

Frame matroids

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 2 & 0 & 3 & 0 & 0 \\ -1 & 1 & 0 & 7 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

≤ 2 nonzero entries per column

Frame matroids

$$
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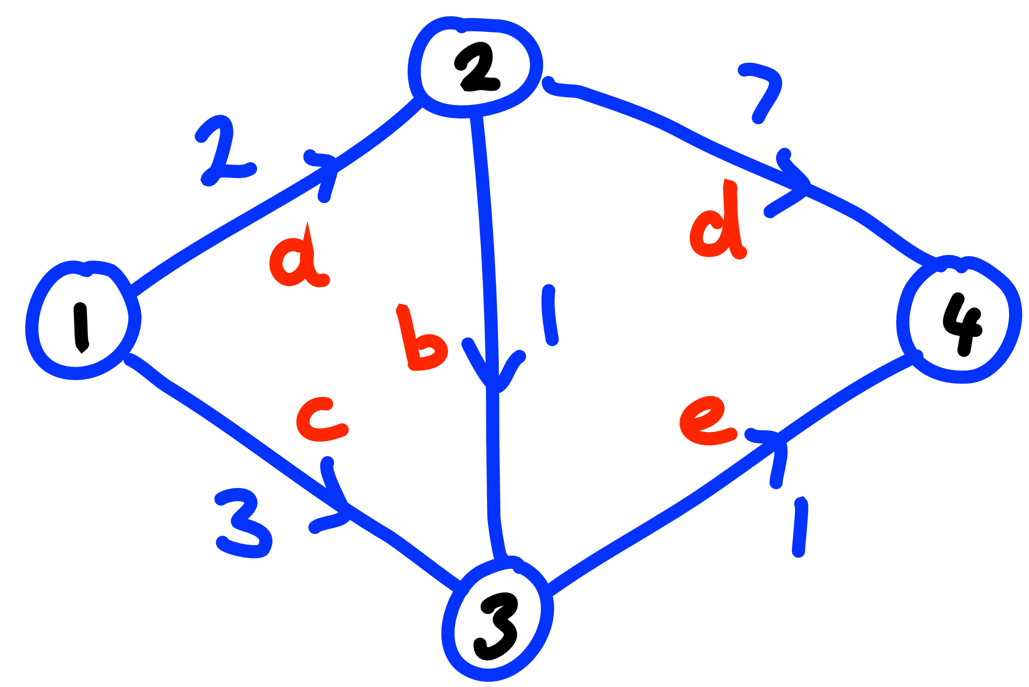
\end{array}
\longrightarrow
$$



group-labelled graph

Perturbation

$$M(A) \longrightarrow M(A+P), \quad \mathrm{rank}(P) = t$$

rank-t perturbation

Structure Theorem [Geelen, Gerards, Whittle]

There exist $k, t \in \mathbb{Z}$ such that:
If $M \in \mathcal{M}$ is vertically k-connected, then there is a rank-t perturbation N of M such that either

(i) N is a frame matroid,

(ii) N^* is a frame matroid, or

(iii) N is representable over some proper subfield of $\mathbb{F}$.

Binary Matroid Structure Theorem

Let $\mathcal{M}$ be a proper minor-closed class of binary matroids. Then there exist $k, t \in \mathbb{Z}$ such that, if $M \in \mathcal{M}$ is vertically k-connected, then there is a rank-t perturbation of M that is either graphic or cographic.

Question: Can we say more?

"Main Theorem"

There exist structurally defined classes $\mathcal{M}_1, \dots, \mathcal{M}_t$ such that:

(i) $\mathcal{M}_1, \dots, \mathcal{M}_t \subseteq \mathcal{M}$, and

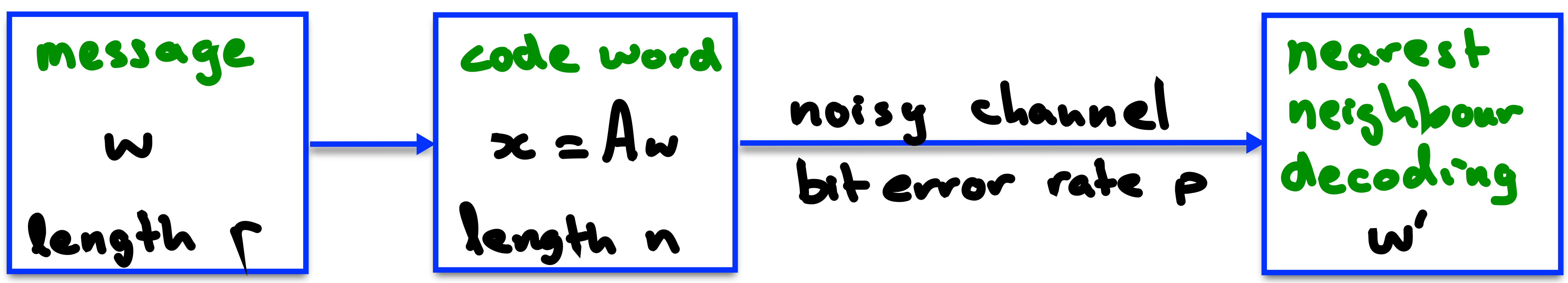
(ii) for some $k \in \mathbb{Z}$, all vertically k-connected matroids in $\mathcal{M}$ are in $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_t$.

Overview

- Coding theory
- Growth rates
- Main theorem
- More on growth rates

Binary linear codes

$A \in GF(2)^{r \times n}$.



Code: $C = \text{rowspace}(A)$

Dimension: r

Length: n

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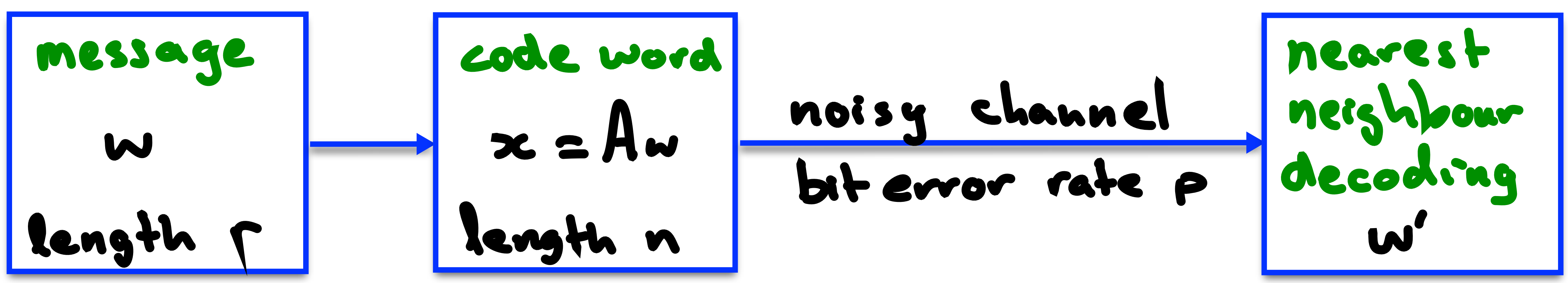

Code: $C = \text{rowspace}(A)$

Dimension: r

Length: n

Matroid: $M(C) = M(A)$

Distance: $\text{girth}(M(C)^*)$

What were they thinking?

C is graphic $\Leftrightarrow$ $M(C)$ is cographic.

Asymptotically good classes

A class $\mathcal{C}$ of codes is asymptotically good if there exists $\varepsilon > 0$ such that for each $R \in \mathbb{Z}$ there exists $C \in \mathcal{C}$ with

- $r(C) \geq R$,
- $\dfrac{r(C)}{\text{length}(C)} \geq \varepsilon$, and (rate)
- $\dfrac{\text{dist}(C)}{\text{length}(C)} \geq \varepsilon$. (relative distance)

Theorem [Van Lint]

The set of binary linear codes is asymptotically good.

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Conjecture

No proper minor-closed class of binary linear codes is asymptotically good.

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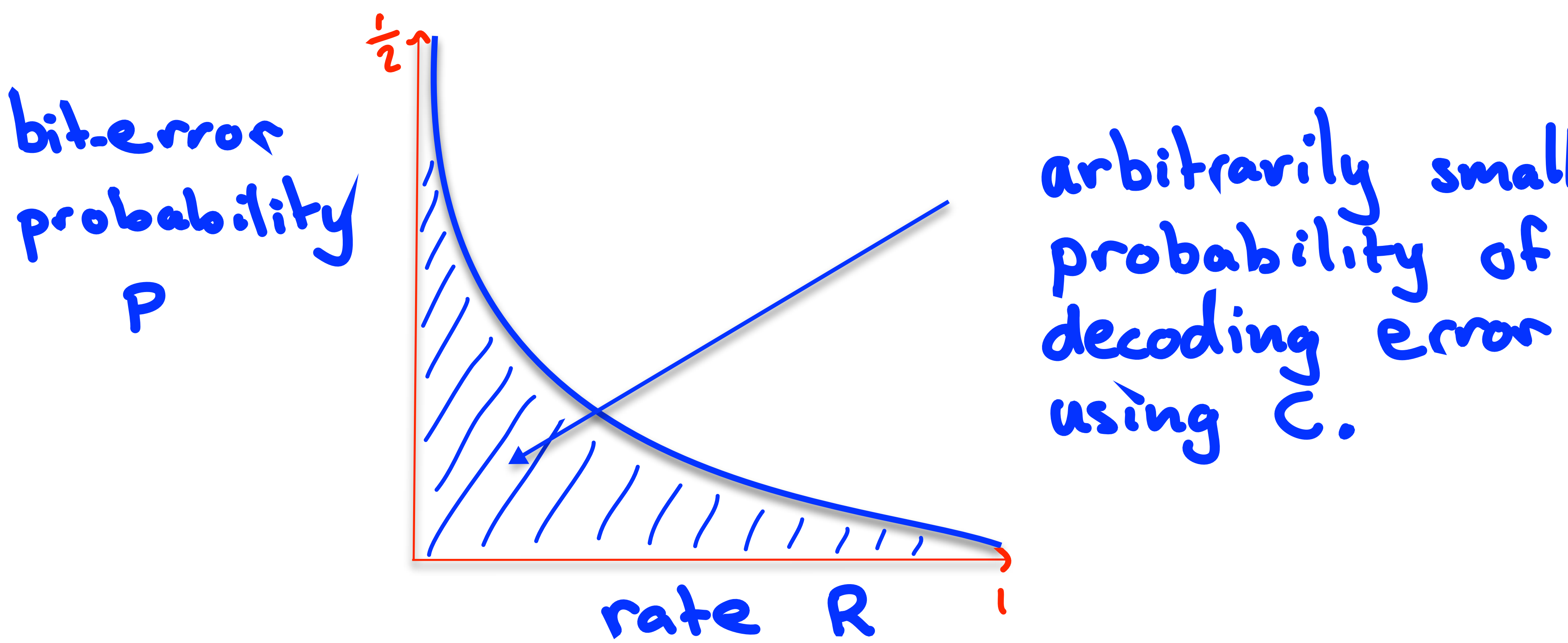
~~**Conjecture**~~ **Theorem** [Nelson, Van Zwam]

No proper minor-closed class of binary linear codes is asymptotically good.

Threshold functions

For a class $\mathcal{C}$ of codes we define $\theta : (0,1) \to [0, \frac{1}{2}]$ such that, for $R \in (0,1)$, $\theta(R)$ is the supremum of the bit-error rate $p \in [0, \frac{1}{2}]$ for which there exist codes of rate $\geq R$ in $\mathcal{C}$ with arbitrarily small error probabilities.



Shannon's Theorem

For the set of all binary linear codes,

$\Theta(R) = f^{-1}(R)$ where

$f(p) = 1 + p\log_2 p + (1-p)\log_2(1-p)$.

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Theorem [Decreusefond, Zemor]

For the set of graphic codes,

$$\theta(R) = \frac{(1-\sqrt{R})^2}{2(1-R)}$$

Shannon's Theorem

For the set of all binary linear codes,

$\Theta(R) = f^{-1}(R)$ where $\Theta(0.1) \cong 0.31$

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Theorem [Decreusefond, Zemor]

For the set of graphic codes,

$$\theta(R) = \frac{(1-\sqrt{R})^2}{2(1-R)} \qquad \theta(0.1) \cong 0.22$$

Lemma

For the set of cographic codes,

$$\theta(R) = 0.$$

Conjecture

For any proper minor-closed class $\mathcal{C}$ of binary linear codes, either

I $\quad \Theta(R) = 0, \quad$ or

II $\quad \Theta(R) = \dfrac{(1-\sqrt{R})^2}{2(1+R)}$ and $\mathcal{C}$ contains all graphic codes.

Binary Matroid Structure Theorem

Let $\mathcal{M}$ be a proper minor-closed class of binary matroids. Then there exist $k, t \in \mathbb{Z}$ such that, if $M \in \mathcal{M}$ is vertically k-connected, then there is a rank-t perturbation of M that is either graphic or cographic.

Growth-rates of minor-closed classes of graphs

Growth rates

For a class $\mathcal{G}$ of graphs we let

$$g(n) := \max\left(|E(G)| \; : \; G \in \mathcal{G} \text{ simple } n\text{-vertex} \right)$$

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Examples

all graphs: $\quad g(n) = \binom{n}{2}$

planar graphs: $\quad g(n) = 3n - 6, \quad n \geq 3$

Mader's Theorem [1967]

For any proper minor-closed class of graphs $g(n) = O(n)$.

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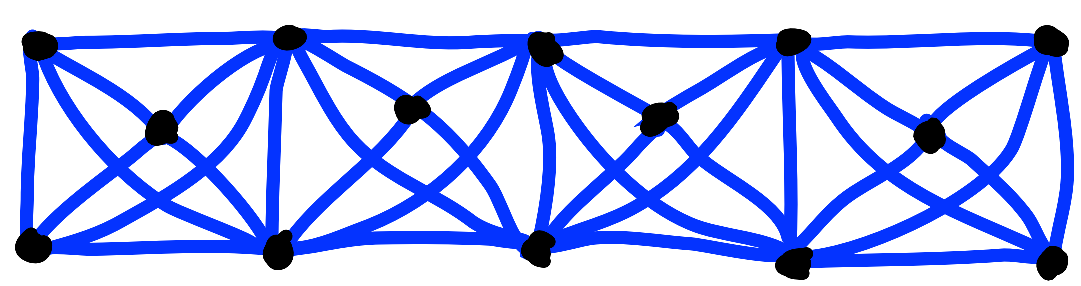
Question: Can we say more?

Example

For the class of graphs with no $K_{3,3}$-minor, if $n \geqslant 3$,

$$g(n) = \begin{cases} 3n-5, & n \equiv 2 \pmod 3 \\ 3n-6, & \text{otherwise} \end{cases}$$



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Conjecture [Norin]

The growth-rate function for any proper minor-closed class of graphs is "eventually periodically-linear".

Growth-rates of minor-closed classes of matroids

Growth rates

For a class $\mathcal{M}$ of matroids we let

$$g(r) := \max\left(|M| : M \in \mathcal{M} \text{ simple rank-}r \right).$$

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Examples

$\mathbb{F}$-rep. matroids: $|\mathbb{F}| = q$ $\quad g(r) = \dfrac{q^r - 1}{q - 1}$

graphic matroids: $\quad g(r) = \binom{r+1}{2}$

frame matroids: $\quad g(r) = (q-1)\binom{r}{2} + r$

cographic matroids: $\quad g(r) = 3(r-1)$

Growth-rate Theorem [Geelen, Kabell, Kung, Whittle]

Let $\mathbb{F}$ be a finite field and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids. Then either:

(i) $g(r) = O(r)$,

(ii) $g(r) = O(r^2)$ and $\mathcal{M}$ contains all graphic matroids, or

(iii) $g(r) = O(q^r)$ and $\mathcal{M}$ contains all $\mathbb{F}'$-rep. matroids where $\mathbb{F}'$ is a q-element subfield of $\mathbb{F}$.

Question: Can we say more?

Let $\mathbb{F}$ be a finite field and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids

Linearly dense

$g(r) = O(r)$

Quadratically dense

$g(r) = O(r^2)$ and $\mathcal{M}$ contains all graphic matroids

Base-q exponentially dense

$g(r) = O(q^r)$ and $\mathcal{M}$ contains all $\mathbb{F}'$-rep. matroids where $\mathbb{F}'$ is a q-element subfield of $\mathbb{F}$.

Linearly dense classes

Conjecture

If $\mathcal{M}$ is linearly dense, then the growth-rate function is eventually periodically linear.

Exponentially dense classes

Theorem [Geelen, Nelson]

If $\mathcal{M}$ is base-q exponentially dense, then there exist $k, d \in \mathbb{Z}$ such that

$$g(r) \approx \frac{q^{r+k}-1}{q-1} - d.$$

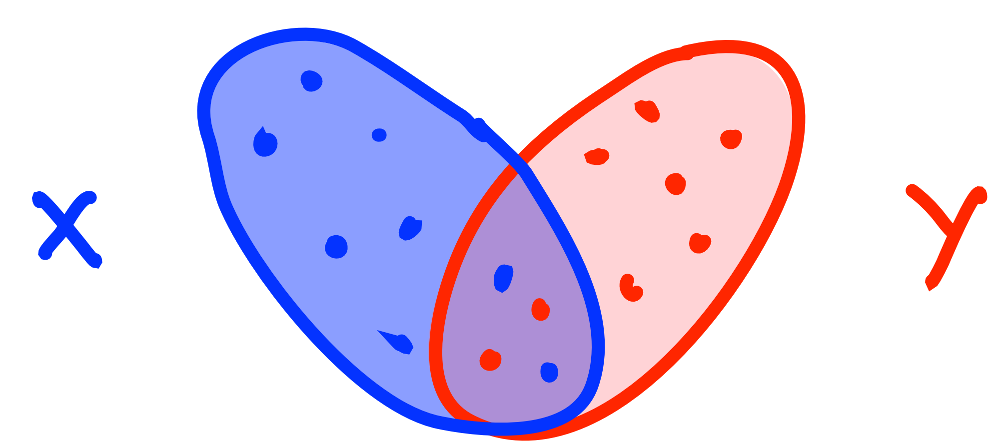
Exponentially dense classes

Lemma If $\mathcal{M}$ is base-q exponentially dense where $q \geq 3$. Then $\mathcal{M}$ contains extremal matroids with arbitrarily large rank that are <u>round</u>.

Exponentially dense classes

Lemma If M is base-q exponentially dense where $q \geq 3$. Then M contains extremal matroids with arbitrarily large rank that are round.

A matroid $M = (E, r)$ is round if for each partition (X, Y) of E either X or Y is spanning.



Quadratically dense classes

Conjecture:
If $\mathcal{M}$ is quadratically dense, then there exist $a, b, c \in \mathbb{Z}$ such that

$$g(r) \approx a\binom{r}{2} + br + c.$$

Quadratically dense classes

Lemma [Geelen, Nelson]
Let $f(r) = a\binom{r}{2} + br + c$.
If $g(r) > f(r)$ infinitely often, then, for each $k \in \mathbb{Z}$, there exist vertically k-connected matroids $M \in \mathcal{M}$ with $|M| > f(r(M))$ and with arbitrarily large rank.

Question: What is the structure of the matroids in $\mathcal{M}$ with high vertical connectivity?

Structure Theorem [Geelen, Gerards, Whittle]

There exist $k, t \in \mathbb{Z}$ such that:
If $M \in \mathcal{M}$ is vertically k-connected, then there is a rank-t perturbation N of M such that either

(i) N is a frame matroid,
(ii) N^* is a frame matroid, or
(iii) N is representable over some proper subfield of $\mathbb{F}$.

Subfield template

$$\Phi = (\mathbb{F}_0, C, D, Y, A_1, A_2, \Delta, \nabla)$$

- $\mathbb{F}_0$ is a subfield of $\mathbb{F}$
- C, D, Y disjoint finite sets
- $A_1 \in \mathbb{F}^{D \times C}$, $A_2 \in \mathbb{F}^{D \times Y}$
- Δ is a subspace of $\mathbb{F}_0^D$ and ∇ is a subspace of $\mathbb{F}_0^{C \cup Y}$.

$$\Phi = (\mathbb{F}_0, C, D, Y, A_1, A_2, \Delta, \Lambda)$$

$$A = \begin{array}{cc} & \begin{array}{ccc} C & Y & \phantom{\text{columns from } \Lambda} \end{array} \\ D & \left[\begin{array}{c|c|c} A_1 & A_2 & \text{columns from } \Lambda \\ \hline \multicolumn{2}{c|}{\begin{array}{c}\text{rows} \\ \text{from} \\ \Delta\end{array}} & \begin{array}{c}\text{entries from} \\ \mathbb{F}_0\end{array} \end{array}\right] \end{array}$$

$$M(\Phi) = \{ M(A)/C \}$$

Frame template

$\Phi = (\Gamma, C, D, X, Y_0, Y_1, A_1, \Delta, \Lambda)$

- Γ is a subgroup of $\mathbb{F}^\times$
- C, D, X, Y_0, Y_1 disjoint finite sets
- $A_1 \in \mathbb{F}^{(D \cup X) \times (C \cup Y_0 \cup Y_1)}$
- Λ is a subgroup of $\mathbb{F}^D$ and

  Δ is a subgroup of $\mathbb{F}^{C \cup Y_0 \cup Y_1}$

$$\Phi = (\Gamma, C, D, X, Y_0, Y_1, A_1, \Delta\ \ \Lambda)$$

$$A' = \begin{array}{cc}
 & \begin{array}{cccc} X & \qquad\qquad\qquad & \qquad\qquad & Y_0\ Y_1\ C \end{array} \\
\begin{array}{c} \\ D \\ \\ \end{array} & \left[\begin{array}{c|c|c|c}
I & 0 & 0 & A_1 \\ \cline{2-2}
 & \text{columns from } \Lambda & & \\ \hline
0 & \Gamma\text{-frame matrix} & \underbrace{\text{unit columns}}_{Z} & \text{rows from } \Delta
\end{array}\right]
\end{array}$$

Get A from A' by adding Y_1-columns to Z-columns.

$$M(\Phi) = \{ M(A) \setminus Y_1 / C \}$$

Main Theorem

There exist subspace templates $\Omega_1, \dots, \Omega_a$, and frame templates $\Phi_1, \dots, \Phi_b$, $\Psi_1, \dots, \Psi_c$ such that

- $\mathcal{M}$ contains $\mathcal{M}(\Omega_1), \dots, \mathcal{M}(\Omega_a), \mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_b)$, $\mathcal{M}(\Psi_1)^*, \dots, \mathcal{M}(\Psi_c)^*$, and
- there exists $k \in \mathbb{Z}$ such that, if $M \in \mathcal{M}$ is vertically k-connected, then M is contained in one of $\mathcal{M}(\Omega_1), \dots, \mathcal{M}(\Omega_a), \mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_b)$, $\mathcal{M}(\Psi_1)^*, \dots, \mathcal{M}(\Psi_c)^*$.

Growth-rates revisited

Representation over two fields

Let $\mathcal{M}(\mathbb{F}_1, \mathbb{F}_2)$ denote the class of matroids representable over both $\mathbb{F}_1$ and $\mathbb{F}_2$.

Problem Determine the growth-rate function for $\mathcal{M}(\mathbb{F}_1, \mathbb{F}_2)$ when $\mathbb{F}_1$ is finite.

Growth-rate Theorem $\Rightarrow$

I. If $\mathrm{char}(\mathbb{F}_1) \neq \mathrm{char}(\mathbb{F}_2)$, then $M(\mathbb{F}_1, \mathbb{F}_2)$ is quadratically dense.

II. If $\mathrm{char}(\mathbb{F}_1) = \mathrm{char}(\mathbb{F}_2)$, then $M(\mathbb{F}_1, \mathbb{F}_2)$ is base-q exponentially dense where q is the size of the largest common subfield of $\mathbb{F}_1$ and $\mathbb{F}_2$.

Conjecture [Kung 1991]

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be fields such that

(i) $|\mathbb{F}_1|$ is finite,

(ii) $\mathrm{char}(\mathbb{F}_1) \neq \mathrm{char}(\mathbb{F}_2)$, and

(iii) the largest common subgroup of $\mathbb{F}_1^\times$ and $\mathbb{F}_2^\times$ has size α.

Then
$$g(r) = \alpha\binom{r}{2} + O(r).$$

Conjecture

For $\mathcal{M}(\mathbb{F}, \mathbb{R})$ where $|\mathbb{F}|$ is finite,

$$g(r) = \begin{cases} \binom{r}{2} + O(r), & |\mathbb{F}| \text{ even} \\ 2\binom{r}{2} + O(r), & |\mathbb{F}| \text{ odd} \end{cases}$$

Conjecture [Kung 1991]

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be fields such that

(i) $|\mathbb{F}_1|$ is finite,

(ii) $\mathrm{char}(\mathbb{F}_1) \neq \mathrm{char}(\mathbb{F}_2)$, and

(iii) $\mathbb{F}_1^\times$ is a subgroup of $\mathbb{F}_2^\times$.

Then $$g(r) \approx (|\mathbb{F}_1|-1)\binom{r}{2} + r.$$

Conjecture

For $\mathcal{M}(\mathbb{F}, \mathbb{C})$ where $|\mathbb{F}|$ is finite,

$$g(r) \approx (|\mathbb{F}|-1)\binom{r}{2} + r.$$

Thank you