Linear relations for a generalized Tutte polynomial

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Why Do Americans Stink at Math?

By **ELIZABETH GREEN** JULY 23, 2014

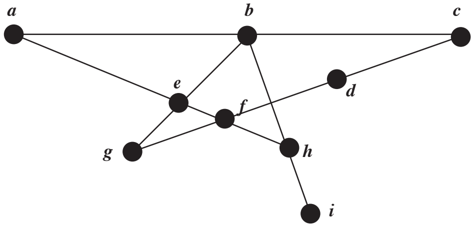
When Akihiko Takahashi was a junior in college in 1978, he was like most of the other students at his university in suburban Tokyo. He had a vague sense of wanting to accomplish something but no clue what that something should be. But that spring he met a man who would become his mentor, and this relationship set the course of his entire career.

Takeshi Matsuyama was an elementary-school teacher, but like a small number of instructors in Japan, he taught not just young children but also college students who wanted to become teachers. At the university-affiliated elementary school where Matsuyama taught, he turned his classroom into a kind of laboratory, concocting and trying out new teaching ideas. When Takahashi met him, Matsuyama was in the middle of his boldest experiment yet — revolutionizing the way students learned math by radically changing the way teachers taught it.

Instead of having students memorize and then practice endless lists of equations — which Takahashi remembered from his own days in school —

Warm-up problem

September 2013 Jungle Gym: Problem 297.



For each point P, record the number of 3-point lines through P, the number of 4-point lines through P, and so on.

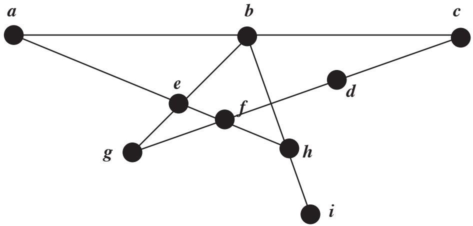
Point	a	b	c	d	...
Address	43	3^3	43	4	...

Question: Is there a finite set of points in the plane where each point has a unique address?

Warm-up problem

Matroid Problem: Find a rank 3 geometry M (represented over the reals) with the property that

- $M - x \ncong M - y$ for all $x \neq y$, and
- $M/x \ncong M/y$ for all $x \neq y$.



$M/a \cong M/c$.

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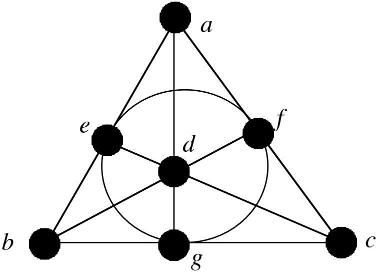


Reid's "Fano" Matroid.

Tutte polynomial

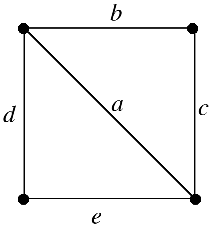
Definition

Let M be a matroid with rank function r. The Tutte polynomial is

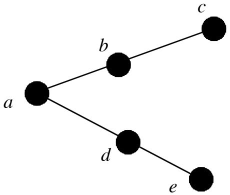
$$T(M; x, y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)} (y-1)^{|A|-r(A)}.$$

Example

$$r(A) = \max_{F \subseteq A}\{|F| \mid F \text{ is acyclic}\}$$



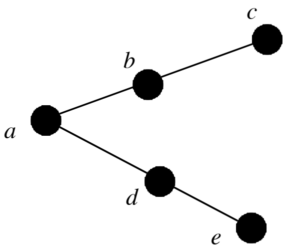
Graph



Matroid

Subset	$\emptyset$	singletons	pairs	triples	4 elts.	E
Rank	0	1	2	2 or 3	3	3

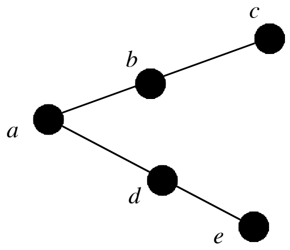
$$T(M;x,y) = \sum_{A\subseteq S} (x-1)^{r(S)-r(A)}(y-1)^{|A|-r(A)}.$$



Subset	$\emptyset$	singletons	pairs	triples	4 elts.	E
Rank	0	1	2	2 or 3	3	3

$A = \{a,b,c\}, r(A) = 2, r(S) = 3 \Rightarrow$ Term is $(x-1)(y-1)$

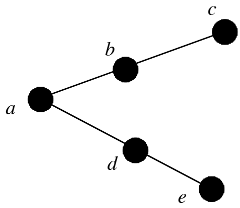
$$T(M; x, y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)} (y-1)^{|A|-r(A)}.$$



Subset	$\emptyset$	singletons	pairs	triples	4 elts.	E
Rank	0	1	2	2 or 3	3	3

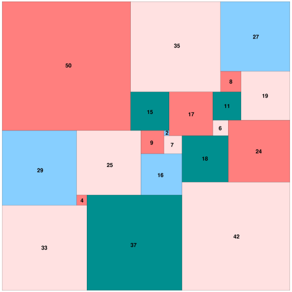
$$T(M; x, y) = x^3 + 2x^2 + 2xy + x + y^2 + y$$

Key recursion: $T(M) = T(M/e) + T(M - e)$



x	y	Interpretation
1	1	# spanning forests (bases)
2	1	# forests (independent sets)
2	0	# acyclic orientations
1	0	# acyclic orientations with unique specified source.

William Thomas Tutte (1917-2002)



[Observation - 1940's] Let $b(G)$ be the number of spanning trees of G. Then

$$b(G) = b(G/e) + b(G - e)$$

$T(M; x, y) = \sum b_{i,j} x^i y^j$, where $b_{i,j}$ is the number of bases of M of internal activity i and external activity j.

Properties

- Fundamental deletion-contraction recursion:
  - $T(M) = T(M - p) + T(M/p)$ if p is neither a loop nor an isthmus (coloop)
  - $T(M) = xT(M/p)$ if p is an isthmus;
  - $T(M) = yT(M - p)$ if p is a loop.
- Dual matroid: $T(M^*; x, y) = T(M; y, x)$.
- Factors: $T(M_1 \oplus M_2) = T(M_1)T(M_2)$.

Free samples

$$T(M; x, y) = \sum b_{i,j} x^i y^j$$

- $x^3 + 2x^2 + 2xy + x + y^2 + y$
- $x^3 + x^2y + 2x^2 + 2xy + 3x + y^3 + 3y^2 + 3y$
- $x^2 + 3x + y^3 + 2y^2 + 3y$
- $x^4 + 3x^3 + 4x^2 + 2x + 2x^2y + 5xy + 2y + xy^2 + 3y^2 + y^3$

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Things you should notice:

1. $b_{i,j} \geq 0$ for all i, j.

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Beta invariant

$T(M_1; x, y) = x^2 y$ $T(M_2; x, y) = x^2 + x + y$

Which matroid is connected?

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Theorem (Crapo '67)

A matroid M (on more than one point) is connected if and only if $b_{1,0} > 0$.

Free samples

$$T(M; x, y) = \sum b_{i,j} x^i y^j$$

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Things Brylawski (1972) noticed:

1. $b_{0,2} + b_{2,0} = b_{1,0} + b_{1,1}$

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Theorem (Brylawski '72)

M is a matroid on n-element set with Tutte polynomial

$$T(M;x,y) = \sum b_{i,j} x^i y^j.$$

For all $0 \leq k < n$,

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} b_{i,j} = 0.$$

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Theorem (C. Merino, A. de Mier, M. Noy, 2001)

A matroid is connected if and only if its Tutte polynomial is irreducible over $\mathbb{Z}[x, y]$.

Recall: $T(M_1 \oplus M_2) = T(M_1)T(M_2)$.

Generalizations

Definition

A *ranked set* is a set S with a rank function r. We write $G = (S, r)$, where the function $r : S \to \mathbb{Z}$ satisfies

(R0) $r(\emptyset) = 0$ [*normalization*]

(R1) $r(A) \leq r(S)$ for all $A \subseteq S$ [*rank S maximum*]

(R2) $r(A) \leq |A|$ for all $A \subseteq S$ [*subcardinality*]

Ranked sets include matroids, antimatroids and greedoids.

$$T(G; x, y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)} (y-1)^{|A|-r(A)} = \sum b_{i,j} x^i y^j$$

Greedoids

Definition

A greedoid G is a pair (S, r) where S is a finite set and $r : 2^S \to \mathbb{Z}^+ \cup \{0\}$ such that:

R0. $r(\emptyset) = 0$ [Normalization]

R1. $r(A) \leq r(A \cup \{p\})$ [Increasing]

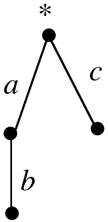
R2. $r(A) \leq |A|$ [Subcardinal]

R3′ If $r(A) = r(A \cup \{p_1\}) = r(A \cup \{p_2\})$, then $r(A \cup \{p_1, p_2\}) = r(A)$. [Local semimodularity]

A greedoid example

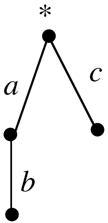
Rooted graphs are greedoids.
Let $S =$ edge set, and define rank as follows:

$$r(A) = \max_{B \subseteq A}\{|B| : B \text{ is a rooted subtree}\}.$$



Subset	$\emptyset$	a	b	c	ab	ac	bc	abc
Rank	0	1	0	1	2	2	1	3

Greedoid Tutte polynomial



$$T(G; x, y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)} (y-1)^{|A|-r(A)}.$$

Deletion-contraction:

$$T(G) = (x-1)^{r(G)-r(G-p)} T(G-p) + T(G/p)$$

Subset	$\emptyset$	a or c	b	ab or ac	...
Rank	0	1	0	2	...
Term	$(x-1)^3$	$(x-1)^2$	$(x-1)^3(y-1)$	$(x-1)$	...

$$T(G; x, y) = x((x-1)^2 y + x) = x^3 y - 2x^2 y + x^2 + xy.$$

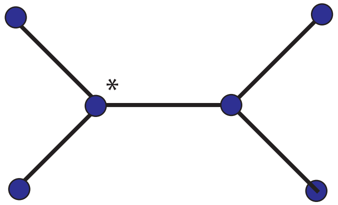
Greedoid Tutte results

$$T(G;x,y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)}(y-1)^{|A|-r(A)} = \sum b_{i,j} x^i y^j$$

Recursion: $T(G) = (x-1)^{r(G)-r(G-p)} T(G-p) + T(G/p)$

Theorem (G. - McMahon ('89))

Let G be a greedoid. Then If G is a rooted tree, then $T(G)$ uniquely determines G.



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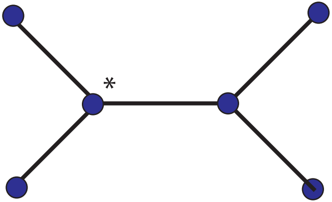


Let $a(T)$ = the number edges of the rooted subtree T, and $b(T)$ = the number of leaves.

Corollary: Rooted trees can be reconstructed from the ordered pairs $\{(a(T), b(T))\}$.

Main result

Theorem

$G = (S, r)$ is a ranked set with $|S| = n$ and Tutte polynomial

$$T(M; x, y) = \sum b_{i,j} x^i y^j.$$

1. *For all $0 \leq k < n$,*

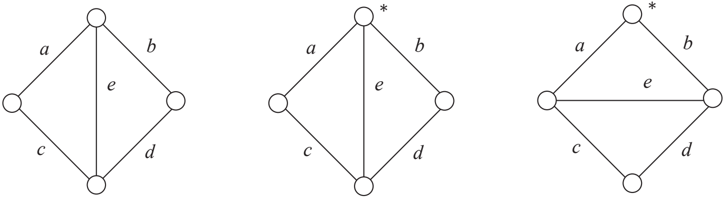
$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} b_{i,j} = 0.$$

2. *For $k = n$,*

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} b_{i,j} = (-1)^{n-r(S)}.$$

Non-matroid examples

$$T(G;x,y)=\sum_{A\subseteq S}(x-1)^{r(S)-r(A)}(y-1)^{|A|-r(A)}=\sum b_{i,j}x^iy^j$$



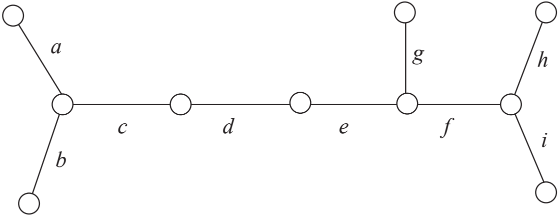
G G' G''

$$\begin{aligned}T(G)&=x^3+2x^2+2xy+x+y^2+y\\T(G')&=x^3y^2-3x^2y^2+2x^2y+x^2+3xy^2-2xy+3x+3y\\T(G'')&=x^3y^3-3x^2y^3+2x^2y+3xy^3-3xy+4x-y^3+y^2+4y\end{aligned}$$

Trees

Rank: $r(A) = \max_{F \subseteq A}\{|F| \mid F \text{ is feasible}\}$.

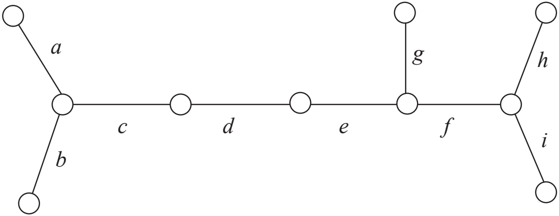
- Feasible sets are subtree complements.
- Convex sets are the subtrees.



Trees

[$k = 1$] Let T be a tree with m interior edges and let $\mathcal{S}$ be the collection of all subtrees with exactly one interior edge. Then

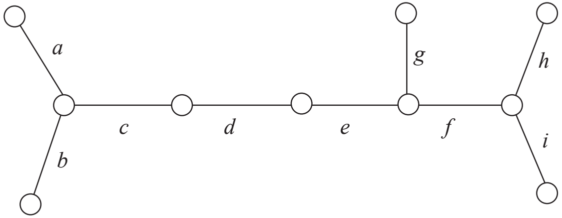
$$\sum_{S \in \mathcal{S}} (-1)^{|S|} = -m.$$



Subtree size	3	4	5
Number	9	6	1

$$-9 + 6 - 1 = -4.$$

Trees



[$k = 2$]

$$(-a_{1,0} + 3a_{2,0} - 6a_{3,0}) + (-3a_{3,1} + 4a_{4,1} - 5a_{5,1}) + (a_{4,2} - a_{5,2} + a_{6,2})$$

	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$a_{i,0}$	1	9	11	3	0	0	0
$a_{i,1}$	0	0	0	9	6	1	0
$a_{i,2}$	0	0	0	0	6	5	1

$$(-9 + 33 - 18) + (-27 + 24 - 5) + (6 - 5 + 1) = 0.$$

Antimatroids

Let $G = (S, r)$ be an antimatroid with $a_{i,j}$ convex sets of size i and interior of size j.

1. For $k < n$,

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} \sum_{s=i}^{n} (-1)^{s-i} \binom{s}{i} a_{s,j} = 0.$$

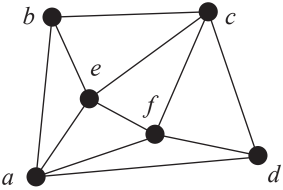
2. For $k = n$,

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{s=i}^{n} (-1)^{s-i} \binom{s}{i} a_{s,j} = 1.$$

Finite subsets $\mathbb{R}^n$

Rank: $r(A) = \max_{F \subseteq A}\{|F| \mid F \text{ is feasible}\}$.

- C is convex if $Hull(C) \cap S = C$.
- Feasible sets are convex set complements.



- Convex: $\{b, c, d, e, f\}$ Not convex: $\{a, b, d\}$

Theorem

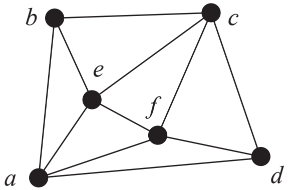
G be an antimatroid with convex sets $\mathcal{C}$. Then

$$T(G; x, y) = \sum_{C \in \mathcal{C}} (x-1)^{|C|} y^{|int(C)|}.$$

Finite subsets $\mathbb{R}^n$

[k=1] Let S be a finite subset of $\mathbb{R}^n$. Let $\mathcal{C}_1$ be the collection of all convex sets with exactly one interior point. Then

$$\sum_{C \in \mathcal{C}_1} (-1)^{|C|} = (-1)^n |int(S)|.$$

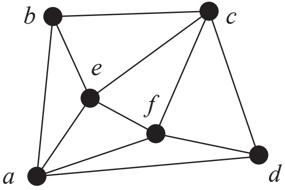


e unique interior	*abce*	*abef*	*abcef*
f unique interior	*acdf*	*adef*	*acdef*

$$b_{0,1} = 4(-1)^4 + 2(-1)^5 = 2 = |int(S)|.$$

Finite subsets $\mathbb{R}^n$

[$k = 2$] $$\sum_{i=0}^{n}(-1)^i\left(\binom{i+1}{2}a_{i,0} + ia_{i,1} + a_{i,2}\right)$$

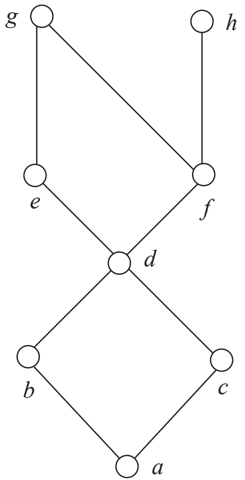


	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$f_i = a_{i,0}$	1	6	15	15	6	1	0
$a_{i,1}$	0	0	0	0	4	2	0
$a_{i,2}$	0	0	0	0	0	1	1

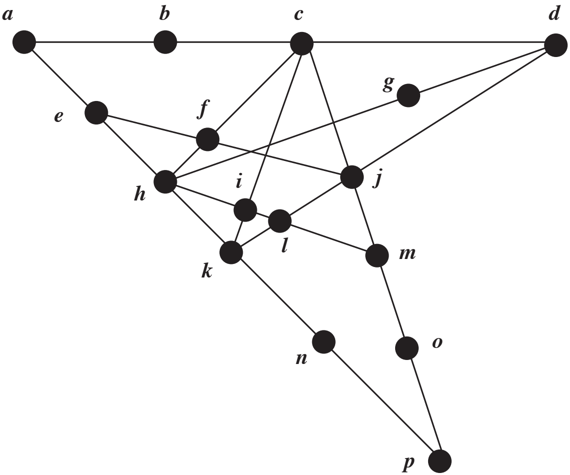
Then $\sum_{i=0}^{n}(-1)^i\left(\binom{i+1}{2}a_{i,0} + ia_{i,1} + a_{i,2}\right) =$
$-(6) + (45) - (90) + (60 + 16) - (15 + 10 + 1) + (1) = 0.$

Other antimatroids and greedoids

- Trees
- Rooted graphs
- Posets
- Chordal graphs
- Subsets of $\mathbb{R}^n$
- ...



Warm-up solution



Conjecture: Almost all matroids have $M - x \ncong M - y$ and $M/x \ncong M/y$ for all x and y.