Linear relations for a generalized Tutte polynomial

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Why Do Americans Stink at Math?

By ELIZABETH GREEN  JULY 23, 2014

When Akihiko Takahashi was a junior in college in 1978, he was like most of the other students at his university in suburban Tokyo. He had a vague sense of wanting to accomplish something but no clue what that something should be. But that spring he met a man who would become his mentor, and this relationship set the course of his entire career.

Takeshi Matsuyama was an elementary-school teacher, but like a small number of instructors in Japan, he taught not just young children but also college students who wanted to become teachers. At the university-affiliated elementary school where Matsuyama taught, he turned his classroom into a kind of laboratory, concocting and trying out new teaching ideas. When Takahashi met him, Matsuyama was in the middle of his boldest experiment yet — revolutionizing the way students learned math by radically changing the way teachers taught it.

Instead of having students memorize and then practice endless lists of equations — which Takahashi remembered from his own days in school —
Warm-up problem

September 2013 Jungle Gym: Problem 297.

For each point $P$, record the number of 3-point lines through $P$, the number of 4-point lines through $P$, and so on.

<table>
<thead>
<tr>
<th>Point</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Address</td>
<td>43</td>
<td>$3^3$</td>
<td>43</td>
<td>4</td>
<td>...</td>
</tr>
</tbody>
</table>

Question: Is there a finite set of points in the plane where each point has a unique address?
Warm-up problem

**Matroid Problem:** Find a rank 3 geometry $M$ (represented over the reals) with the property that

- $M - x \not\cong M - y$ for all $x \neq y$, and
- $M/x \not\cong M/y$ for all $x \neq y$.

$$M/a \cong M/c.$$
Warm-up problem

Matroid Problem: Find a rank 3 geometry $M$ (represented over the reals) with the property that

- $M - x \not\sim M - y$ for all $x \neq y$, and
- $M/x \not\sim M/y$ for all $x \neq y$.

Reid’s “Fano” Matroid.
Tutte polynomial

**Definition**

Let $M$ be a matroid with rank function $r$. The **Tutte polynomial** is

$$T(M; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S) - r(A)} (y - 1)^{|A| - r(A)}.$$
Example

$$r(A) = \max_{F \subseteq A} \{|F| \mid F \text{ is acyclic}\}$$

Graph Matroid

<table>
<thead>
<tr>
<th>Subset</th>
<th>$\emptyset$</th>
<th>singletons</th>
<th>pairs</th>
<th>triples</th>
<th>4 elts.</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2 or 3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
\[ T(M; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S) - r(A)} (y - 1)^{|A| - r(A)}. \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Subset} & \emptyset & \text{singletons} & \text{pairs} & \text{triples} & \text{4 elts.} & E \\
\hline
\text{Rank} & 0 & 1 & 2 & 2 \text{ or } 3 & 3 & 3 \\
\hline
\end{array}
\]

\[ A = \{a, b, c\}, \ r(A) = 2, \ r(S) = 3 \Rightarrow \text{Term is } (x - 1)(y - 1) \]
\[ T(M; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S) - r(A)} (y - 1)^{|A| - r(A)}. \]

\[ T(M; x, y) = x^3 + 2x^2 + 2xy + x + y^2 + y \]

<table>
<thead>
<tr>
<th>Subset</th>
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</table>
Key recursion: \( T(M) = T(M/e) + T(M - e) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td># spanning forests (bases)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td># forests (independent sets)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td># acyclic orientations</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td># acyclic orientations with unique specified source.</td>
</tr>
</tbody>
</table>
[Observation - 1940’s] Let \( b(G) \) be the number of spanning trees of \( G \). Then

\[
b(G) = b(G/e) + b(G - e)
\]

\[
T(M; x, y) = \sum b_{i,j} x^i y^j,
\]

where \( b_{i,j} \) is the number of bases of \( M \) of internal activity \( i \) and external activity \( j \).
Properties

- Fundamental deletion-contraction recursion:
  - $T(M) = T(M - p) + T(M/p)$ if $p$ is neither a loop nor an isthmus (coloop)
  - $T(M) = xT(M/p)$ if $p$ is an isthmus;
  - $T(M) = yT(M - p)$ if $p$ is a loop.

- Dual matroid: $T(M^*; x, y) = T(M; y, x)$.

- Factors: $T(M_1 \oplus M_2) = T(M_1)T(M_2)$. 
Free samples

\[ T(M; x, y) = \sum b_{i,j} x^i y^j \]

- \[ x^3 + 2x^2 + 2xy + x + y^2 + y \]
- \[ x^3 + x^2y + 2x^2 + 2xy + 3x + y^3 + 3y^2 + 3y \]
- \[ x^2 + 3x + y^3 + 2y^2 + 3y \]
- \[ x^4 + 3x^3 + 4x^2 + 2x + 2x^2y + 5xy + 2y + xy^2 + 3y^2 + y^3 \]

Things you should notice:
1. \( b_{i,j} \geq 0 \) for all \( i, j \).
2. \( T(M; 0, 0) = 0 \), i.e., there is no constant term.
3. \( b_{1,0} = b_{0,1} \), i.e., coef. of \( x \) = coef of \( y \).
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Beta invariant

\[ T(M_1; x, y) = x^2 y \]
\[ T(M_2; x, y) = x^2 + x + y \]

Which matroid is connected?
Beta invariant

\[ T(M_1; x, y) = x^2 y \quad \quad T(M_2; x, y) = x^2 + x + y \]

Which matroid is connected?

- \( M_1 \) is a direct sum of a loop and two isthmuses (coloops);
- \( M_2 = U_{2,3} \).
Beta invariant

\[ T(M_1; x, y) = x^2 y \]
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Which matroid is connected?

- \( M_1 \) is a direct sum of a loop and two isthmuses (coloops);
- \( M_2 = U_{2,3} \).

**Theorem (Crapo ’67)**

A matroid \( M \) (on more than one point) is connected if and only if \( b_{1,0} > 0 \).
Free samples

\[ T(M; x, y) = \sum b_{i,j}x^i y^j \]

- \[ x^3 + 2x^2 + 2xy + x + y^2 + y \]
- \[ x^3 + x^2y + 2x^2 + 2xy + 3x + y^3 + 3y^2 + 3y \]
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Things Brylawski (1972) noticed:

1. \[ b_{0,2} + b_{2,0} = b_{1,0} + b_{1,1} \]
Free samples

\[ T(M; x, y) = \sum b_{i,j}x^iy^j \]

- \( x^3 + 2x^2 + 2xy + x + y^2 + y \)
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**Things Brylawski (1972) noticed:**

1. \( b_{0,2} + b_{2,0} = b_{1,0} + b_{1,1} \)
2. \( b_{3,0} + b_{0,2} + b_{1,2} = b_{2,0} + b_{2,1} + b_{0,3} \)
Free samples

\[ T(M; x, y) = \sum b_{i,j} x^i y^j \]

- \( x^3 + 2x^2 + 2xy + x + y^2 + y \)
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Things Brylawski (1972) noticed:

1. \( b_{0,2} + b_{2,0} = b_{1,0} + b_{1,1} \)

2. \( b_{3,0} + b_{0,2} + b_{1,2} = b_{2,0} + b_{2,1} + b_{0,3} \)
Theorem (Brylawski '72)

\( M \) is a matroid on \( n \)-element set with Tutte polynomial

\[
T(M; x, y) = \sum b_{i,j} x^i y^j.
\]

For all \( 0 \leq k < n \),

\[
\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} b_{i,j} = 0.
\]
Theorem (Brylawski ’72)

A matroid is on n-element set with Tutte polynomial

\[ T(M; x, y) = \sum b_{i,j} x^i y^j. \]

For all \( 0 \leq k < n \),

\[
\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} b_{i,j} = 0.
\]

Theorem (C. Merino, A. de Mier, M. Noy, 2001)

A matroid is connected if and only if its Tutte polynomial is irreducible over \( \mathbb{Z}[x, y] \).

Recall: \( T(M_1 \oplus M_2) = T(M_1) T(M_2) \).
A *ranked set* is a set $S$ with a rank function $r$. We write $G = (S, r)$, where the function $r : S \to \mathbb{Z}$ satisfies

1. **Normalization**
   - $(R0)$ \( r(\emptyset) = 0 \)

2. **Rank $S$ maximum**
   - $(R1)$ \( r(A) \leq r(S) \) for all $A \subseteq S$

3. **Subcardinality**
   - $(R2)$ \( r(A) \leq |A| \) for all $A \subseteq S$

Ranked sets include matroids, antimatroids and greedoids.

\[
T(G; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S) - r(A)} (y - 1)^{|A| - r(A)} = \sum b_{i,j} x^i y^j
\]
A greedoid $G$ is a pair $(S, r)$ where $S$ is a finite set and $r : 2^S \rightarrow \mathbb{Z}^+ \cup \{0\}$ such that:

R0. $r(\emptyset) = 0$ [Normalization]

R1. $r(A) \leq r(A \cup \{p\})$ [Increasing]

R2. $r(A) \leq |A|$ [Subcardinal]

R3'. If $r(A) = r(A \cup \{p_1\}) = r(A \cup \{p_2\})$, then $r(A \cup \{p_1, p_2\}) = r(A)$. [Local semimodularity]
A greedoid example

Rooted graphs are greedoids. Let $S =$ edge set, and define rank as follows:

$$r(A) = \max_{B \subseteq A} \{|B| : B \text{ is a rooted subtree}\}.$$
Greedoid Tutte polynomial

\[ T(G; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S) - r(A)} (y - 1)^{|A| - r(A)}. \]

Deletion-contraction:
\[ T(G) = (x - 1)^{r(G) - r(G - p)} T(G - p) + T(G/p) \]

<table>
<thead>
<tr>
<th>Subset</th>
<th>$\emptyset$</th>
<th>$a$ or $c$</th>
<th>$b$</th>
<th>$ab$ or $ac$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>...</td>
</tr>
<tr>
<td>Term</td>
<td>$(x - 1)^3$</td>
<td>$(x - 1)^2$</td>
<td>$(x - 1)^3(y - 1)$</td>
<td>$(x - 1)$</td>
<td>...</td>
</tr>
</tbody>
</table>

\[ T(G; x, y) = x((x - 1)^2 y + x) = x^3 y - 2x^2 y + x^2 + xy. \]
Greedoid Tutte results

\[ T(G; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S) - r(A)} (y - 1)^{|A| - r(A)} = \sum b_{i,j} x^i y^j \]

Recursion: \[ T(G) = (x - 1)^{r(G) - r(G - p)} T(G - p) + T(G/p) \]

Theorem (G. - McMahon ('89))

Let \( G \) be a greedoid. Then if \( G \) is a rooted tree, then \( T(G) \) uniquely determines \( G \).
Greedoid Tutte results

\[ T(G; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S) - r(A)} (y - 1)^{|A| - r(A)} = \sum b_{i,j} x^i y^j \]

Let \( a(T) \) = the number edges of the rooted subtree \( T \), and \( b(T) \) = the number of leaves.

Corollary: Rooted trees can be reconstructed from the ordered pairs \( \{(a(T), b(T))\} \).
Main result

Theorem

\(G = (S, r)\) is a ranked set with \(|S| = n\) and Tutte polynomial

\[ T(M; x, y) = \sum b_{i,j} x^i y^j. \]

1. For all \(0 \leq k < n\),

\[ \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} b_{i,j} = 0. \]

2. For \(k = n\),

\[ \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} b_{i,j} = (-1)^{n-r(S)}. \]
Non-matroid examples

\[ T(G; x, y) = \sum_{A \subseteq S} (x - 1)^{r(S)-r(A)}(y - 1)^{|A|-r(A)} = \sum b_{i,j}x^i y^j \]

\[ T(G) = x^3 + 2x^2 + 2xy + x + y^2 + y \]
\[ T(G') = x^3 y^2 - 3x^2 y^2 + 2x^2 y + x^2 + 3xy^2 - 2xy + 3x + 3y \]
\[ T(G'') = x^3 y^3 - 3x^2 y^3 + 2x^2 y + 3xy^3 - 3xy + 4x - y^3 + y^2 + 4y \]
Trees

Rank: \( r(A) = \max_{F \subseteq A} \{|F| \mid F \text{ is feasible}\} \).

- Feasible sets are subtree **complements**.
- Convex sets are the **subtrees**.

\[
\begin{array}{cccccccc}
  & a & & & & & g & \\
  b & & c & d & e & f & h & i \\
  & & & & & & & \\
\end{array}
\]
Trees

$[k = 1]$ Let $T$ be a tree with $m$ interior edges and let $S$ be the collection of all subtrees with exactly one interior edge. Then

$$\sum_{S \in S} (-1)^{|S|} = -m.$$
Trees

\[ k = 2 \]

\[
(-a_{1,0} + 3a_{2,0} - 6a_{3,0}) + (-3a_{3,1} + 4a_{4,1} - 5a_{5,1}) + (a_{4,2} - a_{5,2} + a_{6,2})
\]

\[
\begin{array}{|c|ccccccc|}
\hline
 & i = 0 & i = 1 & i = 2 & i = 3 & i = 4 & i = 5 & i = 6 \\
\hline
a_{i,0} & 1 & 9 & 11 & 3 & 0 & 0 & 0 \\
a_{i,1} & 0 & 0 & 0 & 9 & 6 & 1 & 0 \\
a_{i,2} & 0 & 0 & 0 & 0 & 6 & 5 & 1 \\
\hline
\end{array}
\]

\[
(-9 + 33 - 18) + (-27 + 24 - 5) + (6 - 5 + 1) = 0.
\]
Antimatroids

Let $G = (S, r)$ be an antimatroid with $a_{i,j}$ convex sets of size $i$ and interior of size $j$.

1. For $k < n$,

$$
\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} \sum_{s=i}^{n} (-1)^{s-i} \binom{s}{i} a_{s,j} = 0.
$$

2. For $k = n$,

$$
\sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{s=i}^{n} (-1)^{s-i} \binom{s}{i} a_{s,j} = 1.
$$
Finite subsets $\mathbb{R}^n$

Rank: $r(A) = \max_{F \subseteq A} \{|F| \mid F \text{ is feasible}\}$.

- $C$ is convex if $\text{Hull}(C) \cap S = C$.
- Feasible sets are convex set complements.

Convex: $\{b, c, d, e, f\}$  Not convex: $\{a, b, d\}$

**Theorem**

$G$ be an antimatroid with convex sets $\mathcal{C}$. Then

$$T(G; x, y) = \sum_{C \in \mathcal{C}} (x - 1)^{|C|} y^{\text{int}(C)}.$$
Finite subsets $\mathbb{R}^n$

Let $S$ be a finite subset of $\mathbb{R}^n$. Let $C_1$ be the collection of all convex sets with exactly one interior point. Then

$$\sum_{C \in C_1} (-1)^{|C|} = (-1)^n |int(S)|.$$
Finite subsets $\mathbb{R}^n$

\[ [k = 2] \quad \sum_{i=0}^{n} (-1)^i \left( \binom{i+1}{2} a_{i,0} + ia_{i,1} + a_{i,2} \right) \]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
& i = 0 & i = 1 & i = 2 & i = 3 & i = 4 & i = 5 & i = 6 \\
\hline
f_i = a_{i,0} & 1 & 6 & 15 & 15 & 6 & 1 & 0 \\
a_{i,1} & 0 & 0 & 0 & 0 & 4 & 2 & 0 \\
a_{i,2} & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
\]

Then \( \sum_{i=0}^{n} (-1)^i \left( \binom{i+1}{2} a_{i,0} + ia_{i,1} + a_{i,2} \right) = -(6) + (45) - (90) + (60 + 16) - (15 + 10 + 1) + (1) = 0. \)
Other antimatroids and greedoids

- Trees
- Rooted graphs
- Posets
- Chordal graphs
- Subsets of $\mathbb{R}^n$
- ...
Conjecture: Almost all matroids have $M - x \not\cong M - y$ and $M/x \not\cong M/y$ for all $x$ and $y$. 