Accessibility in transitive graphs

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Topic

We look for connections between cycle spaces and cut spaces of graphs.

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And in particular:

Question

Are there connections between the cycle space and the cut space of the same graph?

Cut and cycle spaces

Definition

- A cut is the edge set between A and B for a bipartition $\{A, B\}$ of the vertex set. The cut space of a graph is the set of all finite sums (over $\mathrm{GF}(2)$) of finite cuts.
- The cycle space of a graph is the set of all finite sums (over $\mathrm{GF}(2)$) of edge sets of finite cycles.

A first result

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

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Why does it give an answer to our question?

Reformulating Dunwoody's theorem

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

A finitely presented group G has a locally finite Cayley graph Γ whose cycle space is generated by $\{g(C) \mid C \in \mathcal{C}, g \in G\}$ for some finite set $\mathcal{C}$ of cycle space elements, that is, its cycle space is a finitely generated G-module.

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Theorem (Dicks & Dunwoody 1989)

The cut space of a locally finite Cayley graph of a finitely generated accessible group is a finitely generated $\mathrm{Aut}(G)$*-module.*

Reformulating Dunwoody's theorem

Theorem (Dunwoody 1985)

Let G be a locally finite Cayley graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$-module, then so is its cut space.

Main theorem

Definition

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Let G be a quasi-transitive graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$*-module, then so is its cut space.*

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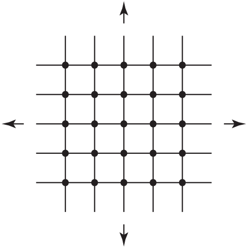
Remark

We cannot ask for an 'if and only if':
Bieri and Strebel (1980) gave an example of a finitely generated accessible group that is not finitely presented.

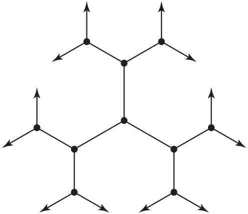
Definition

Two rays, i.e. one-way infinite paths, in a graph G are *equivalent* if for any finite vertex set $S \subseteq V(G)$ both rays lie eventually in the same component of $G - S$. Its equivalence classes are the *ends* of the graph.



one end

two ends



infinitely many ends

Reformulating Dunwoody's thm (once more)

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Theorem (Thomassen & Woess 1993)

A finitely generated group is accessible if and only if one (and hence every) of its locally finite Cayley graphs is accessible.

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Theorem (Dunwoody 1985)

Every locally finite Cayley graph G whose cycle space is a finitely generated $\mathrm{Aut}(G)$-module is accessible.

A conjecture

Conjecture (Diestel 2010)

Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.

A conjecture and an answer

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Theorem (H 2014)

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We obtain a combinatorial proof of

1. Dunwoody's accessibility theorem of finitely presented groups

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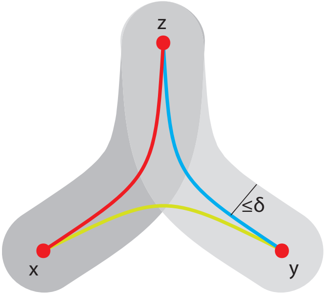
Theorem (Dunwoody 2007)

Every locally finite quasi-transitive planar graph is accessible.

Definition

A connected graph G is called hyperbolic if there exists some $\delta \geq 0$ such that for any three vertices x, y, z of G and for any three shortest paths, one between every two of the vertices, each of those paths lies in the δ-neighbourhood of the union of the other two.



Applications III

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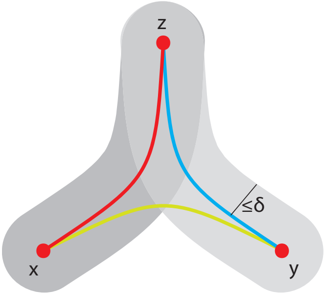

Conjecture (Dunwoody 2011)

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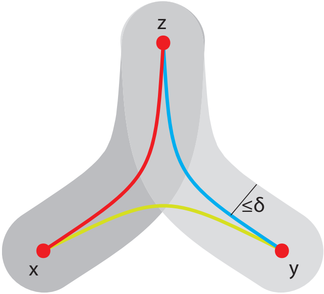

Conjecture (Dunwoody 2011)

Every locally finite transitive hyperbolic graph is accessible.

Theorem (H 2014)

Every locally finite quasi-transitive hyperbolic graph is accessible.

Main theorem

Theorem (H 2014)

Let G be a quasi-transitive graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$-module, then so is its cut space.

Brief sketch of the proof

Let $\mathcal{C}$ be a (possibly infinite) set of finitely many cycles with their $\mathrm{Aut}(G)$-images that generate the cycle space.

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If $\mathcal{E}'$ has *many* orbits, one of them contains never a minimal or maximal element of such a chain.
It can be shown that this is impossible. □

Generalisation to Matroids

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Let M be a connected finitary binary matroid such that finitely many circuits with their images generate every circuit.
Does there exist some finite set of finite cocircuits that generate together with their images every finite cocircuit?

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Answer: No!

Problem

Generalise the main theorem in a suitable way to matroids.