Inequivalent Representations of $\mathbb{F}$-representable Matroids

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Inequivalent Representations

Two $\mathbb{F}$-matrices A and B are *equivalent* if A can be obtained from B by a sequence of the following operations

- Elementary row operations
- Column scaling
- Deleting or adding zero-rows

An Example: $\mathcal{U}_{2,4}$

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \star & \star \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \star \end{bmatrix} \end{array}$$

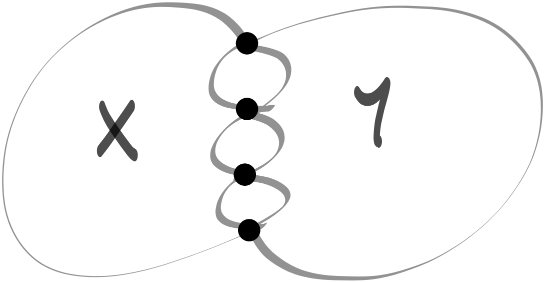
So, $\mathcal{U}_{2,4}$ has $|\mathbb{F}| - 2$ inequivalent representations over $\mathbb{F}$.

Tutte Connectivity

$$\lambda(X) = r(X) + r(E - X) - r(E).$$

$$\lambda(X) = \dim(\langle X \rangle \cap \langle Y \rangle)$$

Tutte Connectivity



$\lambda(X) = 3$

Tutte Connectivity

Definition

A k-**separation** of M is a partition (A, B) of $E(M)$ such that $\lambda(A) \leq k - 1$ and $|A|, |B| \geq k$.

Definition

A matroid is k-**connected** if it does not have any ℓ-separations for $\ell < k$.

Kahn's Conjecture

Conjecture (Kahn '88)

For every finite field $\mathbb{F}$, there exists a constant c such that every 3-connected $\mathbb{F}$-representable matroid has at most c inequivalent representations.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix}$$

Kahn's Conjecture

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Oxley, Vertigan and Whittle '96

False for all $q > 5$!

What about 4-connectedness?

Theorem (Geelen and Whittle, 2013)

For every **prime** *p, there exists a constant c such that every 4-connected $\mathbb{F}_p$-representable matroid has at most c inequivalent representations.*

Geelen, Gerards, Whittle, 2010

False for non-prime fields!

Our Main Theorem

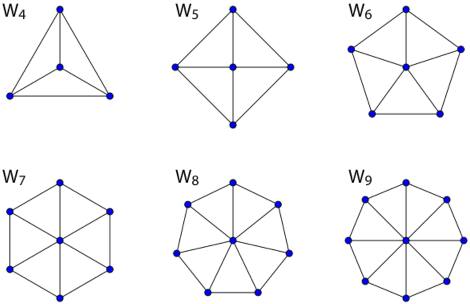
Theorem (Geelen, Gerards, H, van Zwam)

For every finite field $\mathbb{F}$, there exists c and k such that every k-connected matroid has at most c inequivalent representations over $\mathbb{F}$.

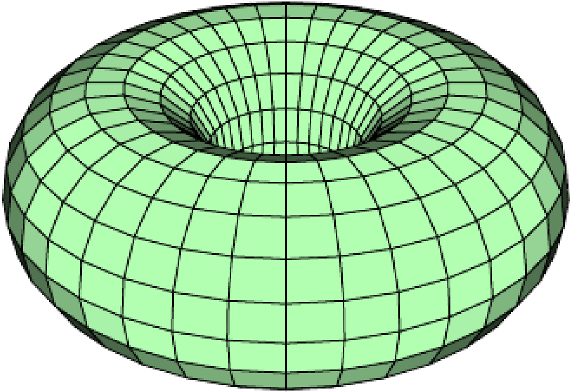
Tutte's Wheels Theorem

Theorem (Tutte)

Let G be a 3-connected graph that is not a wheel. Then G has an element e such that $G \setminus e$ or G/e is **simple** *and 3-connected.*



Fails for 4-connectivity



Internal 4-connectivity

Theorem (Chun, Mayhew, Oxley, 2012)

Let M be an internally 4-connected binary matroid with $|E(M)| \geq 7$. Then M has an internally 4-connected minor N such that $|E(M)| - |E(N)| \leq 3$ unless M or M^* is the cycle matroid of a planar or Möbius ladder, or the cube, or the terrahawk.

The Right Notion

f-Connectivity

Let $f : \{1, \ldots, n\} \to \mathbb{Z}$.
We say that a matroid M is f-**connected** if for all k-separations (A, B) of M, with $k \leq n$, we have $|A| \leq f(k)$ or $|B| \leq f(k)$.

Examples

Connectivity, $f = (0)$
Tutte k-connectivity, $f = (0, 1, \dots, k-2)$.
Internally 4-connectivity $f = (0, 1, 3)$.

An Approximate Chain Theorem

Lemma

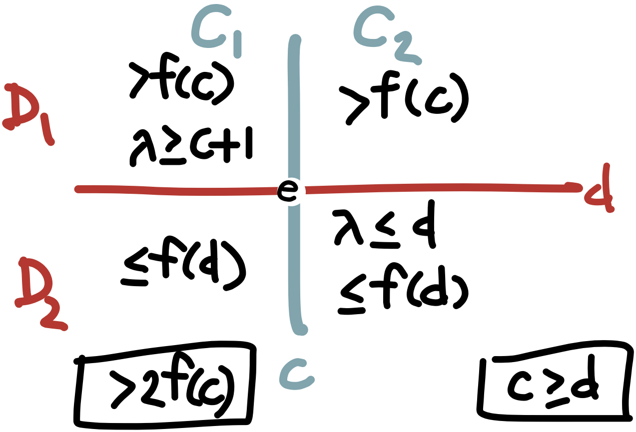
Let M be an f-connected matroid with f non-decreasing. Then for every element $e \in E(M)$, either M/e or $M \setminus e$ is $2f$-connected.

Bixby-Coullard Inequality

If $e \in E(M)$ and (C_1, C_2) and (D_1, D_2) are partitions of $E(M) - e$, then

$$\lambda_{M/e}(C_1) + \lambda_{M\setminus e}(D_1) \geq \lambda_M(C_1 \cap D_1) + \lambda_M(C_2 \cap D_2).$$

Proof



C_1
C_2
D_1
$>f(c)$
$\lambda \geq c+1$
$>f(c)$
e
d
$\lambda \leq d$
D_2
$\leq f(d)$
$\leq f(d)$
$>2f(c)$
c
$c \geq d$

A Chain Theorem for f-connectivity

Theorem

For all $k \in \mathbb{N}$ there exists an $f : \{1, \ldots, k\} \to \mathbb{Z}$ such that if M is a large f-connected matroid and X is a set of elements with $\lambda(X) \leq k$, then there exists elements an f-connected minor N of M such that $X \subseteq E(N)$ and $|E(M)| - |E(N)| \leq 2$.

Fixed and Co-Fixed Elements

Definition

An element e of a matroid M is **fixed** if it is not possible to extend M by an element x' such that $\{x, x'\}$ is an independent clonal pair.

Definition

An element e is **co-fixed** if e is fixed in M^*.

Observation

If M is an $\mathbb{F}$-representable matroid and e is fixed, then a representation of $M \setminus e$ that extends to a representation of M, does so uniquely.

Controlling non-Fixed Elements

Theorem (Geelen and van Zwam)

For every finite field $\mathbb{F}$, there exists a constant k, such that if M is an $\mathbb{F}$-representable f-connected matroid, and X is the set of non-fixed elements of $\mathcal{M}$, then there is a f-small set Y, with $X \subseteq Y$ and $\lambda(Y) \leq k$.

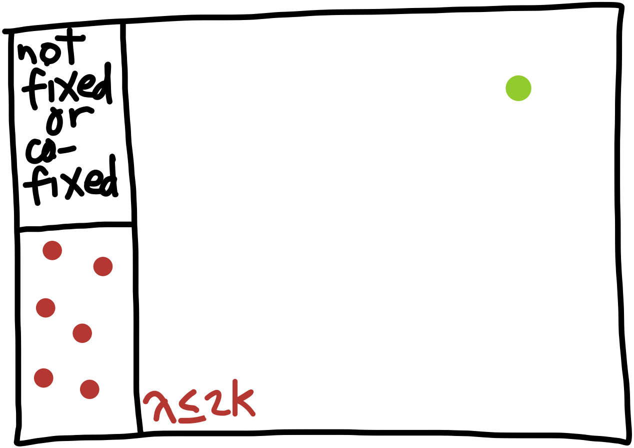
Proof of the Theorem

Theorem (Geelen, Gerards, H, van Zwam)

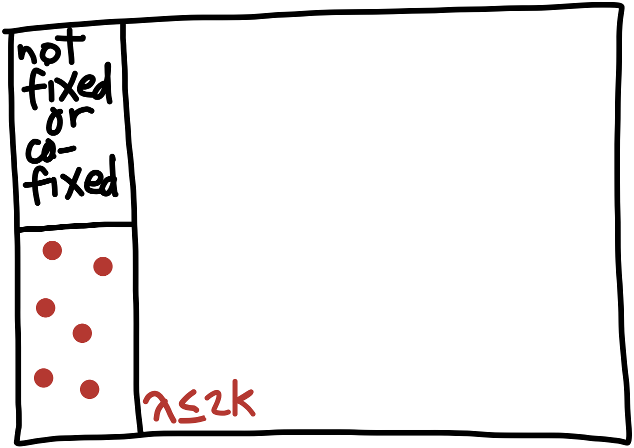
For every finite field $\mathbb{F}$, there exists a constant c and a vector f such that every f-connected matroid has at most c inequivalent representations over $\mathbb{F}$.

Proof. Choose k from the previous theorem and $f : \{1, \ldots, 2k\} \to \mathbb{Z}$ from our Chain Theorem.

Proof of the Theorem



Proof of the Theorem



Proof of the Theorem

- Repeat this procedure.
- Stop when we reach a bounded size matroid M'.
- The number of inequivalent representations of M is at most the number of inequivalent representations of M'.

Thank You!