A Refinement of the Grid Theorem

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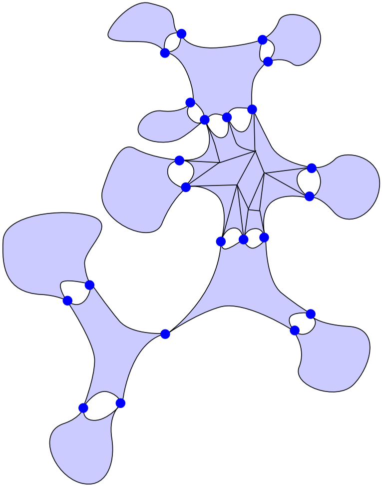
Graphs with no K_5 minor

Theorem (Wagner-1937)

A graph has no K_5-minor if and only if it can be obtained from planar graphs and V_8 by 0-, 1-, 2- *and* 3-*sums.*



Graphs with no K_6-minor

What structure do graphs with no K_6-minor have?

Graphs with no K_6-minor

Conjecture (Jørgensen)

Every 6-connected graph with no K_6-minor is apex (planar plus one vertex).

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Every sufficiently large 6-connected graph with no K_6-minor is apex.

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- Non 6-connected?

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Every graph with no K_6 minor has a "width-5 tree-decomposition" such that each node is "small" or "almost planar."

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Pseudo-Conjecture

Every graph with no K_6 minor has a "width-5 tree-decomposition" such that each node is "small" or "almost planar."

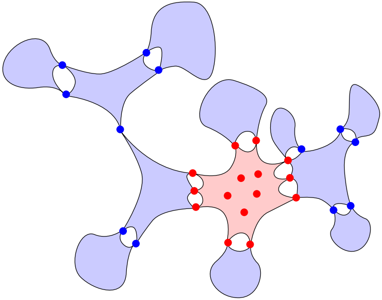
- "small": bounded number of vertices
- "almost planar": planar plus a few vertices

Tree Decomposition

Definition

A *tree-decomposition* of a graph G is a tree T whose nodes each contain a “bag” (subset) of vertices from G such that

- for every vertex v in G, the bags containing v form a non-empty subtree of T and
- for every edge e in G, there is a bag containing both ends of e



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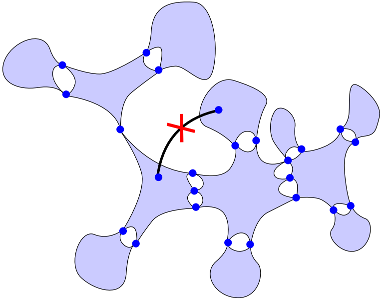
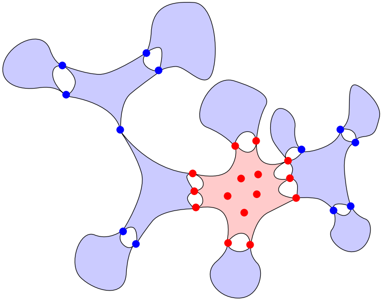


Tree Decomposition

Definition

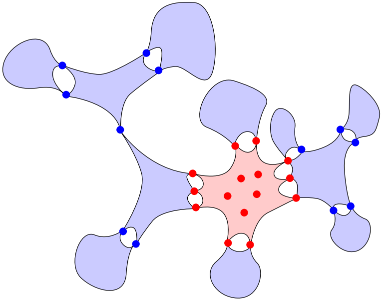
- *Node-width of T* (nw(T)): maximum number of vertices in any bag



Tree Decomposition

Definition

- *Node-width of T* ($\mathrm{nw}(T)$): maximum number of vertices in any bag
- *Tree-width of G* ($\mathrm{tw}(G)$): (minimum node-width of any tree-decomposition of G)-1



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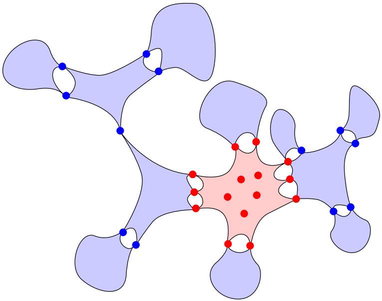
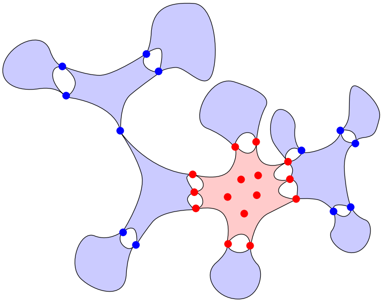


- Small tree-width: decompose the graph along bounded-order separations into small pieces

Tree Decomposition

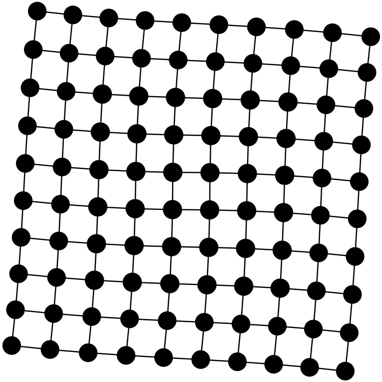


- Small tree-width: decompose the graph along bounded-order separations into small pieces
- Large tree-width: ...?

Grid Theorem

Theorem (Grid Theorem, Robertson and Seymour 1986)

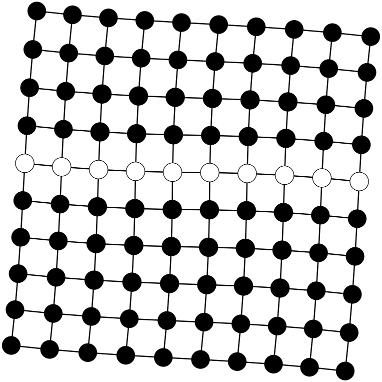
A graph G has large tree-width if and only if G contains a large grid-minor.



k-Connected Set

Definition

A k-connected set in a graph G is a set X of vertices such that for all subsets $Y, Z \subseteq X$ with $|Y| = |Z| \leq k$, G contains $|Y|$ vertex-disjoint paths from Y to Z.



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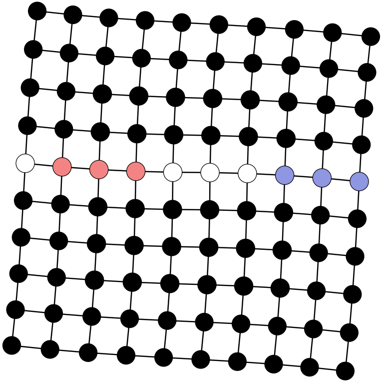

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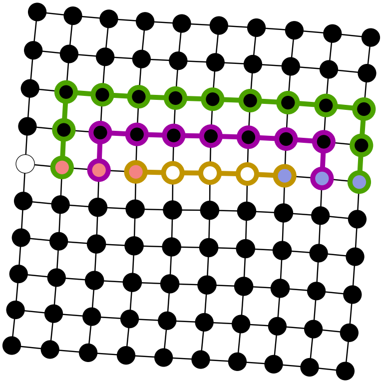

Grid Theorem and Highly-Connected Sets

Lemma (Diestel, Gorbunov, Jensen, Thomassen 1999)

A graph G has large tree-width if and only if it contains a large, highly-connected set.

Grid Theorem and Highly-Connected Sets

Lemma (Diestel, Gorbunov, Jensen, Thomassen 1999)

A graph G has large tree-width if and only if it contains a large, highly-connected set.

Theorem (Equivalent to Grid Theorem)

Every graph with a large highly-connected set contains a large grid-minor.

Tree Decomposition

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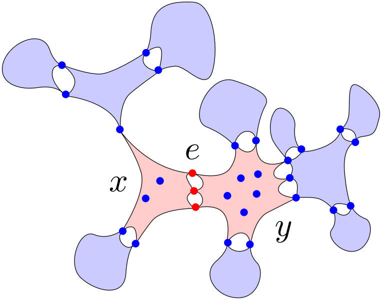
- *Edge-bag of* $e = xy \in E(T)$: intersection of the node-bags for x and y



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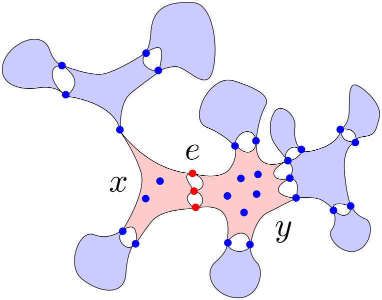
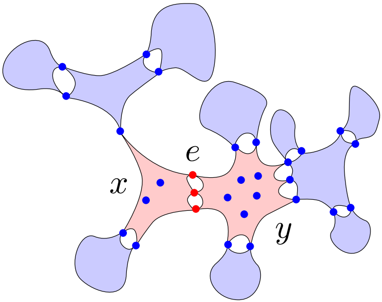


Tree Decomposition

Definition

- *Edge-bag of* $e = xy \in E(T)$: intersection of the node-bags for x and y
- *Edge-width of* T ($\mathrm{ew}(T)$): maximum number of vertices in any edge-bag
- k*-tree-width of* G ($\mathrm{tw}_k(G)$): (minimum node-width of any tree-decomposition T with $\mathrm{ew}(T) \leq k$)-1



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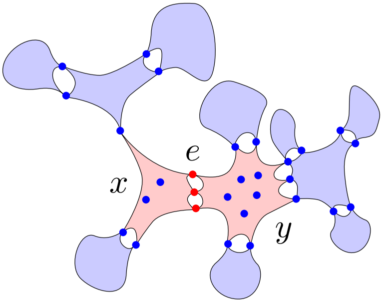

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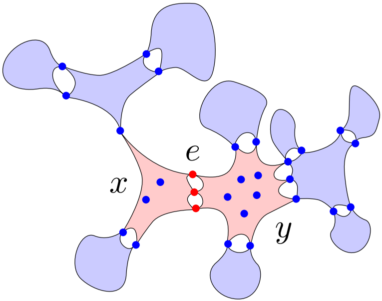
Tree Decomposition



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- Large k-tree-width: ...?

Tree Decomposition

Lemma

$\mathrm{tw}_k(G)$ *is large if and only if* G *contains a large* $(k+1)$*-connected set.*

Tree Decomposition

Lemma

$\mathrm{tw}_k(G)$ is large if and only if G contains a large $(k+1)$-connected set.

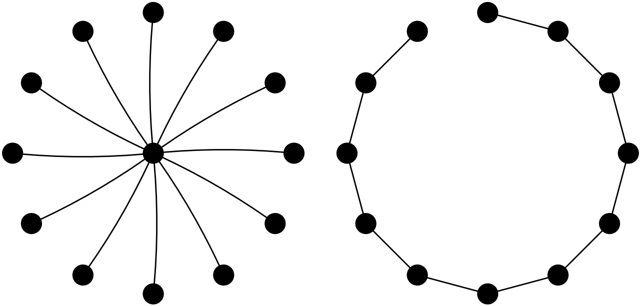
Question

What are the unavoidable minors for graphs with a large k-connected set?

Unavoidable Minors: $k = 1$

Observation

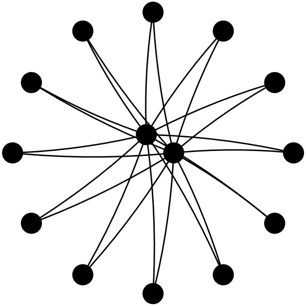
Any graph with a sufficiently large 1*-connected set contains a large star or a long path.*

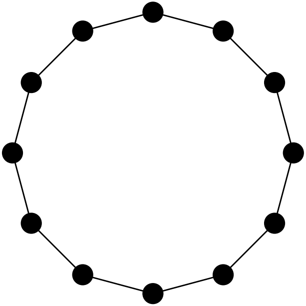


Unavoidable Minors: $k = 2$

Observation

Any graph with a sufficiently large 2-connected set contains a large $K_{2,n}$-minor or a long cycle.

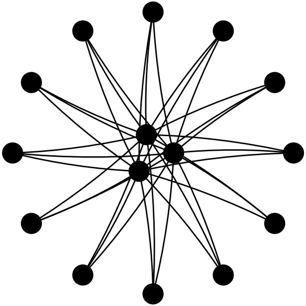




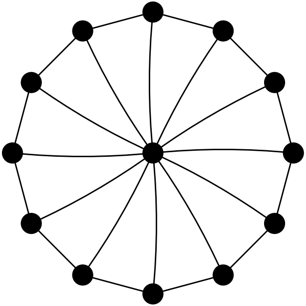
Unavoidable Minors: $k = 3$

Theorem (Oporowski, Oxley, Thomas 1993)

Any graph with a sufficiently large 3-connected set contains a $K_{3,n}$-minor or (length-n)-wheel-minor.



$K_{3,n}$

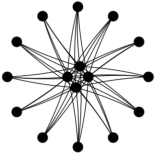


length-n-wheel

Unavoidable Minors: $k = 4$

Theorem (Oporowski, Oxley, Thomas 1993)

Any graph with a sufficiently large 4-connected set contains one of the following minors:



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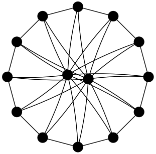

2-hub wheel

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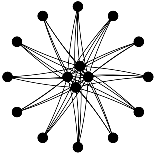

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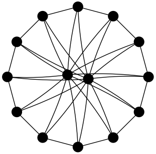

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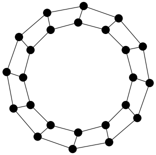

circular ladder

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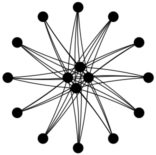

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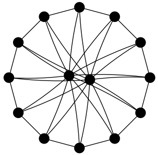

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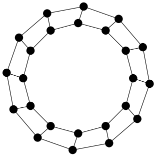

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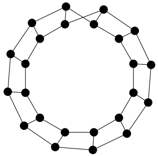

Möbius ladder

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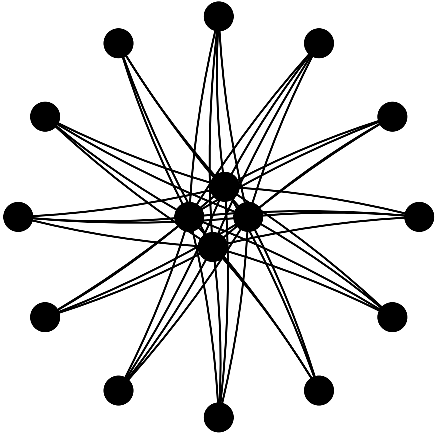

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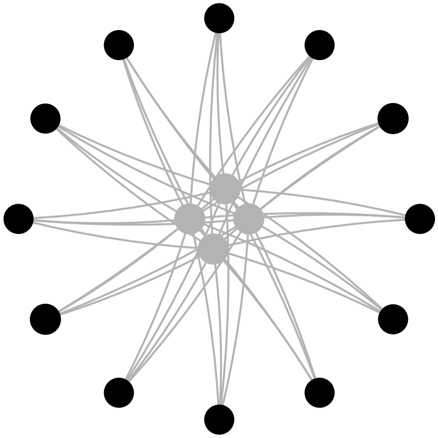

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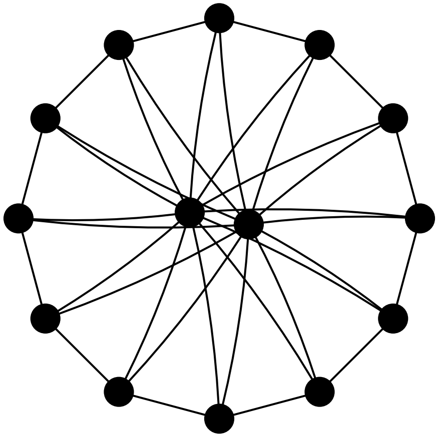

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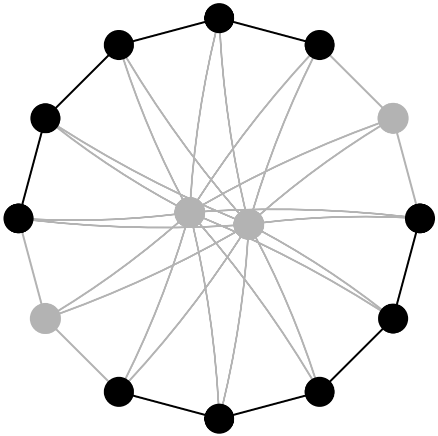

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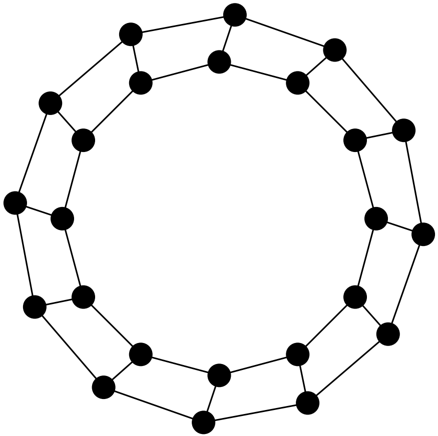

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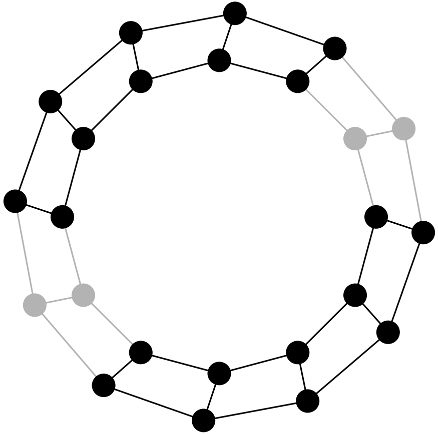

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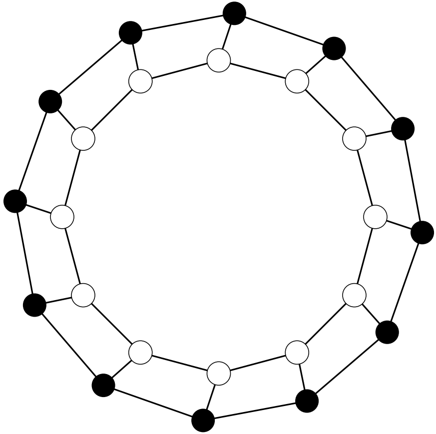

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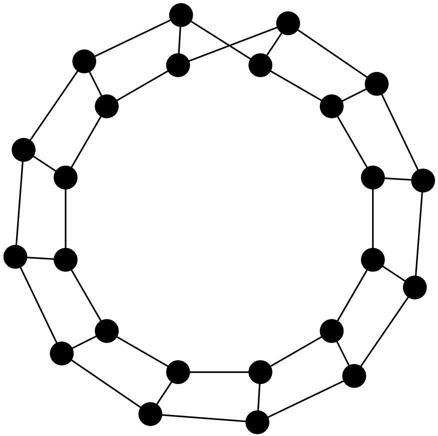

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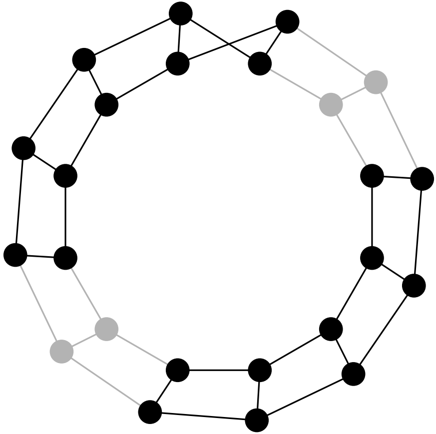

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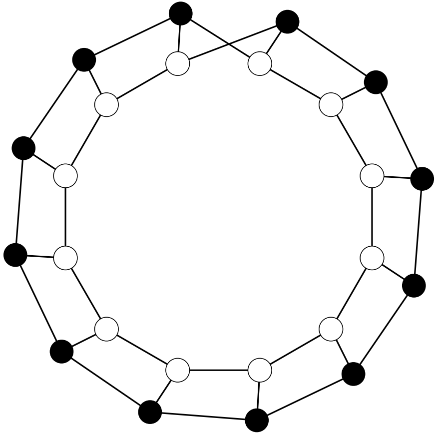

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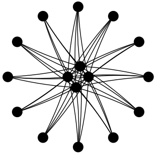

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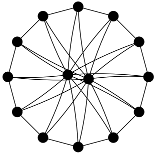
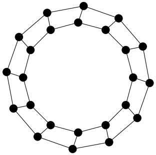
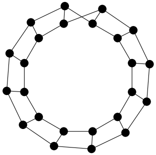


2-hub wheel



circular ladder



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Generalized Wheel

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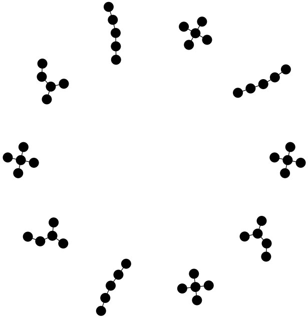

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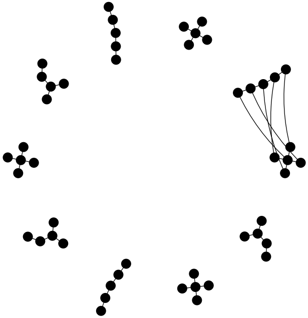

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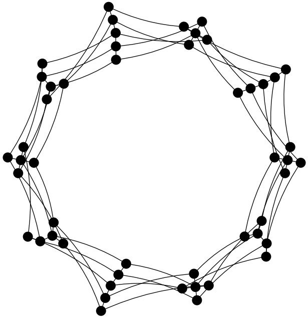

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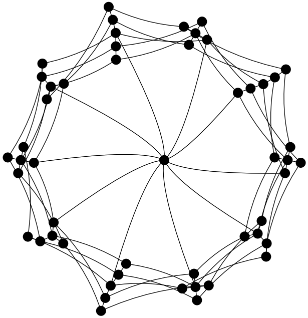

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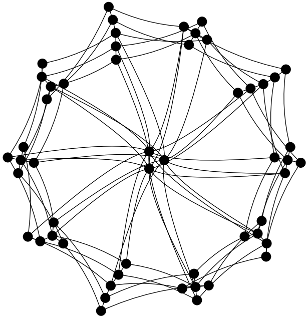

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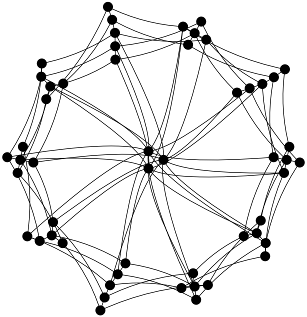
- for each $z \in Z$, an edge from z to T_i for all i

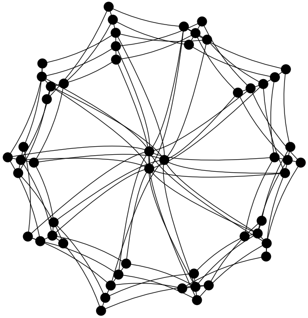


$(5, 3, 10)$-wheel

Unavoidable Minors

Theorem

Any graph with a sufficiently large k-connected set contains a $K_{k,n}$- or (r, ℓ, n)-wheel-minor with $2r + \ell \geq k$.



Unavoidable Minors

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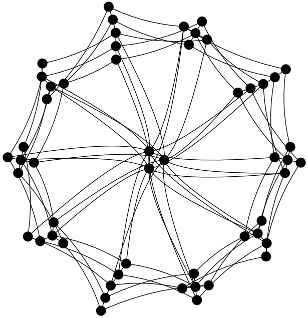
Any graph with a sufficiently large k-connected set contains a $K_{k,n}$- or (r, ℓ, n)-wheel-minor with $2r + \ell \geq k$.

Theorem (Equivalent to Grid Theorem)

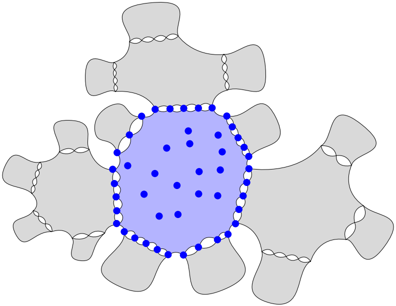
Every graph with sufficiently large, sufficiently highly-connected set contains an large grid-minor.



Graphs with no K_6-minor

Pseudo-Conjecture

Every graph with no K_6-minor has a tree-decomposition T with $\mathrm{ew}(T) \leq 5$ such that each node-bag of T is "small" or "almost planar."



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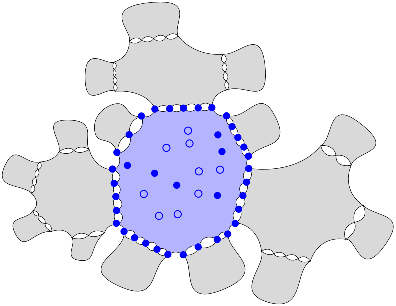

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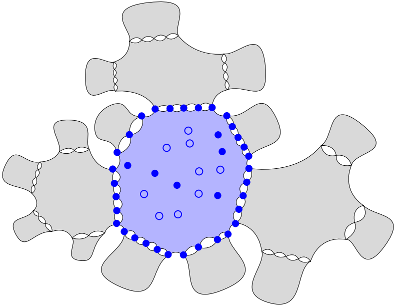
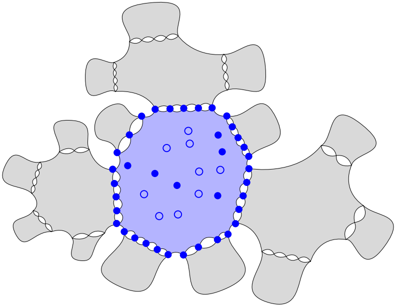


What is the structure of graphs with a large 6-connected set, but no K_6-minor?

Graphs with no K_6-minor



Theorem

Any graph with a sufficiently large 6-connected set contains a $K_{6,n}$-minor or (r, ℓ, n)-wheel-minor with $2r + \ell \geq 6$.

Graphs with no K_6-minor

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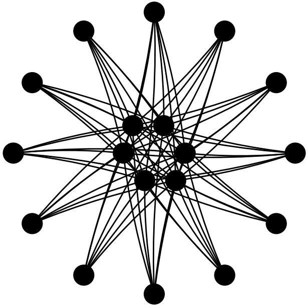
Any graph with a sufficiently large 6-connected set contains one of the following minors:



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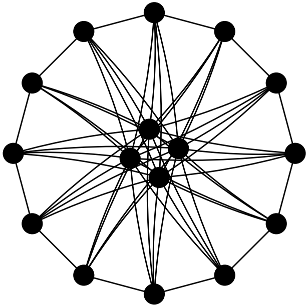

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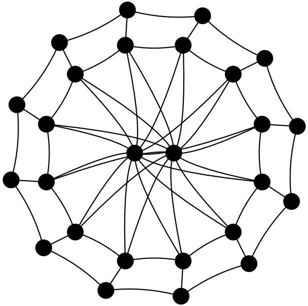

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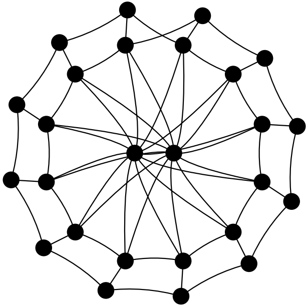

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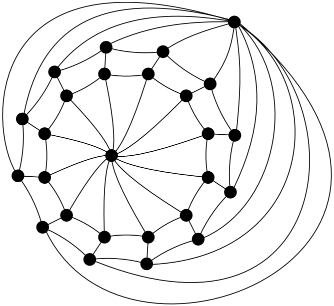

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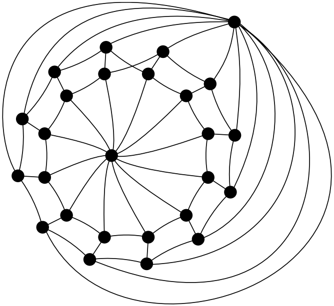

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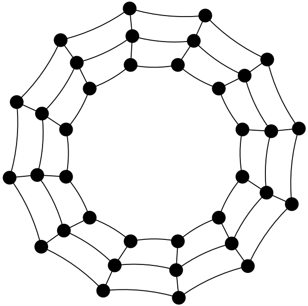

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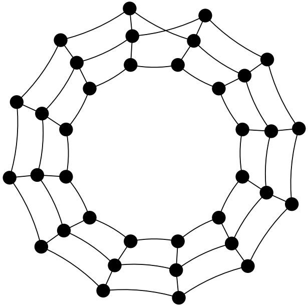

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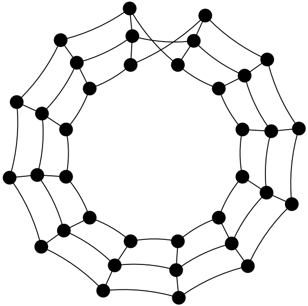

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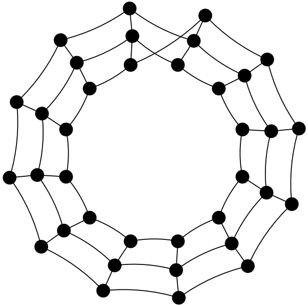

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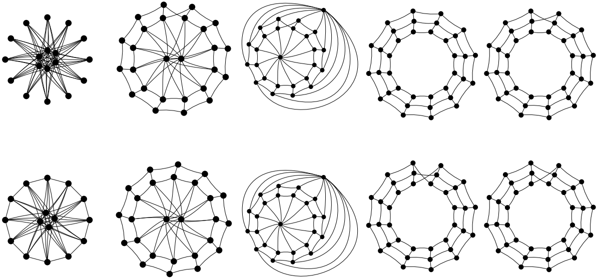

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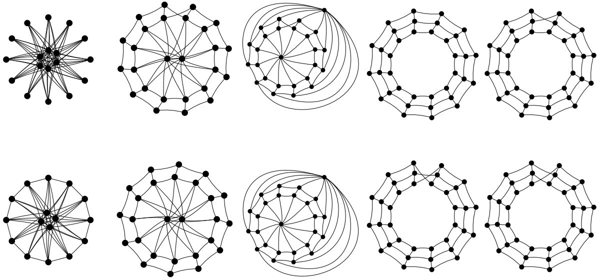

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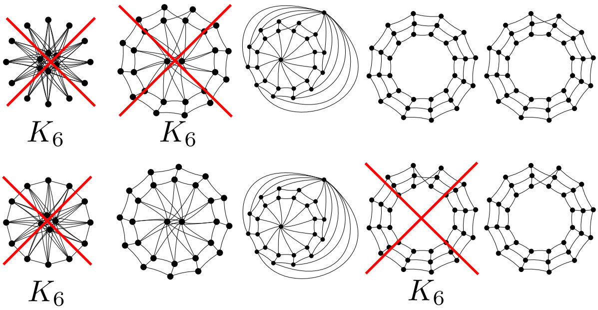

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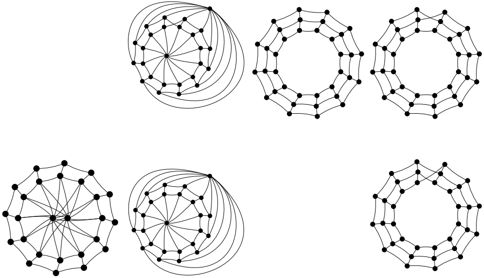

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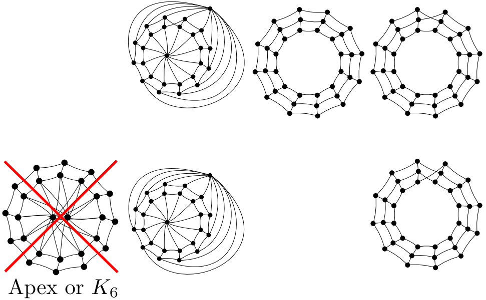

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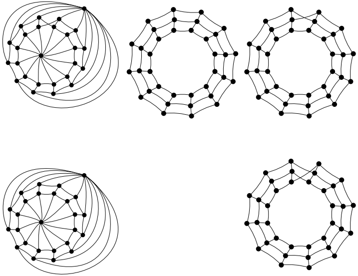

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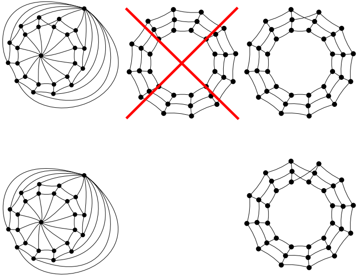

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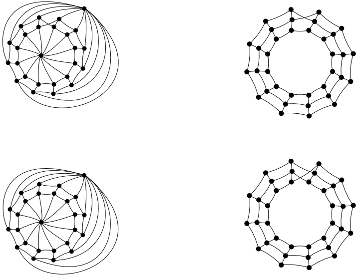

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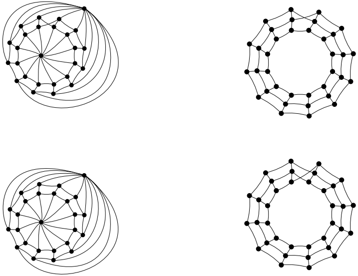
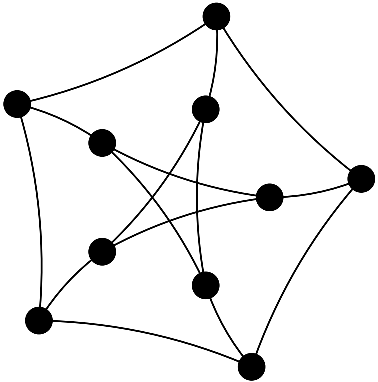


How can these 4 graphs be extended in non-apex ways without creating a K_6-minor?

Graphs with no Petersen graph-minor

What is the structure of graphs with no Petersen graph-minor?



Thank You