Combinatorial limits

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Combinatorial limits and matroid theory

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Combinatorial limits

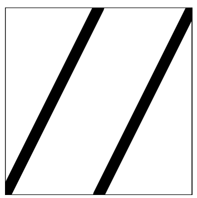
- dense (graph) limits
  well-understood, many applications

- sparse graph limits
  mysterious, many (open) problems

- matroid limits?

Why limits?

- asymptotic properties of large (discrete) objects
  we implicitly use limits in our considerations anyway
- How does the seq. $1, 3, \ldots, 2n-1, 2, 4, \ldots, 2n$ look like?
  How does the adjacency matrix of $K_{n,n}$ look like?
  How does the adjacency matrix of $K_{n,n+1}$ look like?
- convergence of a sequence of discrete objects vs.
  formal analytic representation of its limit



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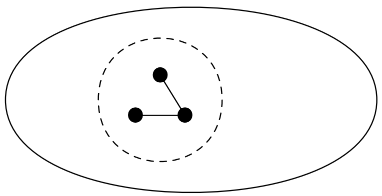
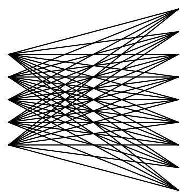
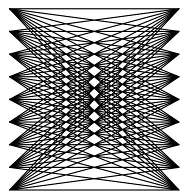
Dense graph convergence

- Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi
- convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $p(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is L-convergent if $p(H, G_n)$ converges for every H
- extendable to other discrete structures

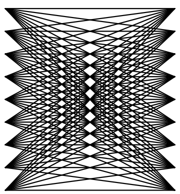




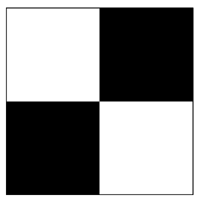


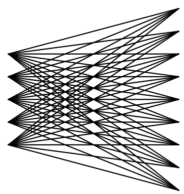
Limit object: graphon

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $p(H, W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $p(H,W) = \lim_{n\to\infty} p(H, G_n)$



$$\begin{matrix} 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0 \end{matrix}$$





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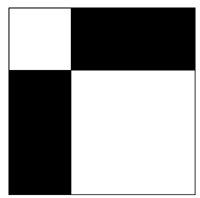

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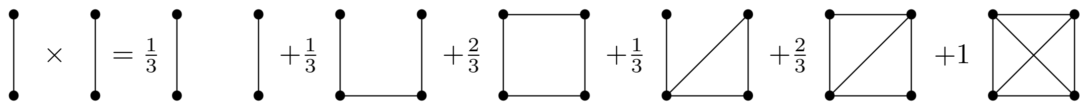
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $p(H,W) = \lim_{n\to\infty} p(H,G_n)$
- every L-convergent sequence of graphs has a limit
- W-random graph converge to W with probability one

Construction of the limit

- sequence of mutually refining regularity partitions
  removal lemma $\Rightarrow$ subgraph counts
- interpret the partitions as functions $[0,1]^2 \to [0,1]$
  the pointwise limit is the sought graphon
  existence by martingale convergence
  uniqueness upto measure preserving permutations

Extremal combinatorics

- graphons capture subgraph densities
  $p(K_{1,2}, W) = 3 \int W(x,y)W(x,z)(1-W(y,z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
- flag algebra method of Razborov (density calculations)



$$p(K_2, W) \times p(K_2, W) = \frac{1}{3} p(K_2 \cup K_2, W) + \frac{1}{3} p(P_4, W) + \cdots$$

- search for inequalities can be computer assisted

$$(\alpha K_2 - \beta \overline{K_2})^2 \geq 0$$

- led to solving problems with no progress for decades

Other connections

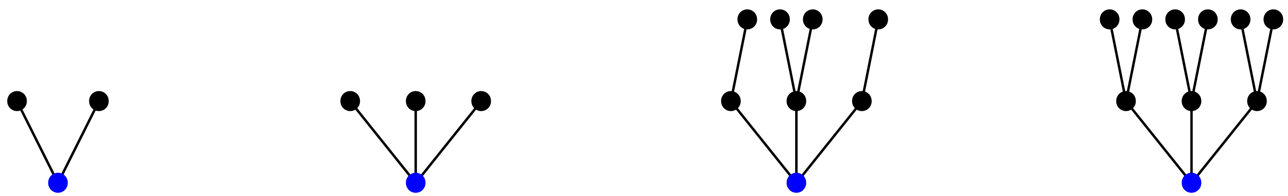
- property and parameter testing
  algorithms using a small sample of a large input
  cover of the space of all graphons
- weak regularity partitions
  cover of the space of typical vertices of a graphon
- quasirandomness
  characterization of quasirandom behavior
  uniqueness of constant and step graphons

Permutation limits

- permutation of order n: order on numbers $1, \ldots, n$
  subpermutation: 4<u>53</u>21<u>6</u> $\longrightarrow$ 213
- probability measure μ on $[0,1]^2$ with unit marginals
  Hoppen, Kohayakawa, Moreira, Ráth and Sampaio
  similar ideas in work of Presutti and Stromquist
- μ-random permutation
  choose n random points, x- and y-coordinates

Sparse graph convergence

- introduced by Benjamini and Schramm'01
- graphs with bounded maximum degree
  trivially L-convergent to $W \equiv 0$
- a sequence $(G_n)_{n\in\mathbb{N}}$ is BS-convergent
  if the distribution of d-neighborhoods converges for all d



More questions than answers

- limit object: graphing (bounded degree Borel graph)
- Does every graphing has a sequence converging to it?
  Aldous and Lyons Conjecture, relation to group theory
- BS-convergence fails to capture global properties
  random cubic graphs vs. random cubic bipartite graphs
  the limit graphing is far from being unique
- local-global convergence (Hatami, Lovász, Szegedy)
  large deviation convergence (Borgs, Chayes, Gamarnik)

First order convergence

- introduced by Nešetřil and Ossona de Mendez'13
- applies to both dense and sparse graphs
  in general any type of relational structures
- FO formula $\varphi(x_1, \ldots, x_k)$ with k free variables
  $\mathbb{P}(\varphi, G)$ probability that k random vertices satisfy φ
- $(G_n)_{n\in\mathbb{N}}$ is FO-convergent if $\forall\varphi$ $\mathbb{P}(\varphi, G_n)$ converges
  subgraph density, existence of a subgraph, etc.

Relation to other notions

- FO-convergence $\Rightarrow$ L-convergence
  $\varphi(x_1, \ldots, x_k)$ says that $x_1, \ldots, x_k$ induce H
- FO-convergence $\Rightarrow$ BS-convergence
  $\varphi(x)$ describing the structure of a d-neighborhood of x
- neither of the converse implications is true
  a convergent sequence of triangle-free graphs
  add a single triangle to every second element

Limit object: modelling

- a modelling M is a graph on G with $V(G) = [0,1]$ every FO-definable subset of $[0,1]^k$ is measurable
- M is a limit of FO-convergent sequence $(G_n)_{n\in\mathbb{N}}$ if $\mathbb{P}(\varphi, G_n) \to \mathbb{P}(\varphi, M)$ holds for every φ
- a modelling M is strong if for every $A, B \subseteq [0,1]$

$$\mu(A) \min_{a\in A} \deg_B(a) \leq \mu(B) \max_{b\in B} \deg_A(b)$$

Modellings vs. other limits

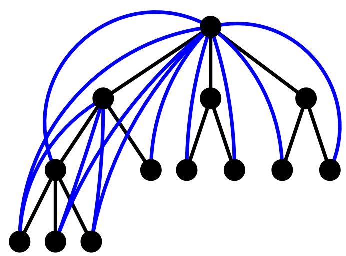
- a modelling of an L-convergent sequence is a graphon
  the graphon is a zero-one graphon
- a strong modelling of a BS-convergent seq. is a graphing
- a graphon need not be a modelling of
  an FO-convergent sequence
- an FO-convergent sequence may have no modelling
  a sequence of $G_{n,1/2}$ is FO-convergent
  graphon uniqueness $\Rightarrow$ non-existence of a modelling

Matroid limits?

- What properties are we interested in?
  independence of elements
  random elements of the limit satisfy that. . .
- good: axiomatization of infinite matroids by Bruhn, Diestel, Kriesell, Pendavingh and Wollan
- bad: matroids of large dense graphs are sparse
  sparse approaches fail due to absence of locality
  first order convergence can offer some help

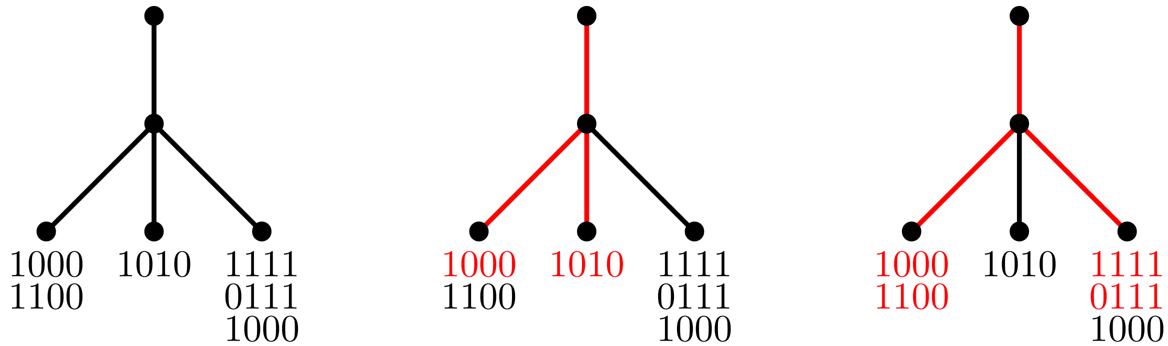
Existence of graph modellings

- Nešetřil and Ossona de Mendez'13
  Every somewhere-dense class of graphs contains an FO-convergent sequence with no modelling.
  Every FO-convergent sequence of graphs with bounded tree-depth has a modelling.
- $\mathrm{td}(G)$ = minimum depth of T with $G \subseteq \overline{T}$
  G has large tree-depth $\Leftrightarrow$ G has a long path



Matroid branch-depth

- analogue of graph tree-depth
- branch-depth of M is the minimum depth of a tree T
  the number of edges of T is the rank of M
  elements of M assigned to leaves of T (one-to-many)
  $r(A) \leq$ #edges of the subtree induced by $A \subseteq M$



Is this a meaningful notion?

- YES because…
- the branch-depth of $M(G)$ is at most $\mathrm{td}(G)$
- the branch-width of M is bounded by its branch-depth
- M has large branch-depth $\Leftrightarrow$ M has a long circuit
- there exists a polynomial-time algorithm that finds a branch-decomposition of an input oracle-given matroid M of depth at most $f(\mathrm{td}(M))$

Existence of matroid modellings

- modelling = infinite matroid on a probability space
- Every FO-convergent sequence of matroids of bounded branch-depth representable over GF(q) has a modelling.
  the limit modelling is representable over GF(q)
- Bad news: no such result exists for
  rank-three matroids representable over rationals
  binary matroids of unbounded branch-depth

Open problems

- Aldous-Lyons conjecture
  every graphing is a limit of a graph sequence
- modellings of graphs with bounded tree-width
  planar graphs? minor-closed classes of graphs?
- robust notion of non-dense convergence
  robust notion of matroid convergence
  global structure, sensitivity to minor changes

Thank you for your attention!