Infinite gammoids

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What are gammoids?

Definition (Edmonds and Fulkerson 1965)

Given a bipartite graph G with vertex classes V and W, let $M_T(G) = (V, \{\text{matchable subsets of } V\})$. A transversal matroid is a matroid isomorphic to $M_T(G)$ for some bipartite graph G.

Definition (Perfect 1968, Mason 1972)

Given a digraph D, and a set $B_0 \subseteq V := V(D)$ of sinks, let $M_L(D, B_0) = (V, \{\text{linkable subsets of } V\})$. A strict gammoid is a matroid isomorphic to $M_L(D, B_0)$ for some dimaze (D, B_0). A gammoid is a matroid restriction of a strict gammoid.

(D, B_0) is called a dimaze; B_0 the exits.

Nice properties of finite gammoids

1. Every finite dimaze defines a strict gammoid. (Perfect 1968, Mason 1972)
2. The class of finite gammoids is minor-closed. (Ingleton and Piff 1973)
3. A finite strict gammoid is dual to a finite transversal matroid, and vice versa. The class of finite gammoids is closed under duality. (Ingleton and Piff 1973)

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Matroid axioms

Set E, $\mathcal{I} \subseteq 2^E$, $\mathcal{I}^{\max}$ = maximal elements in $\mathcal{I}$ with respect to set inclusion

$M = (E, \mathcal{I})$ is a matroid if the following hold:

- (I1) $\emptyset \in \mathcal{I}$.
- (I2) If $I \subseteq I'$ and $I' \in \mathcal{I}$, then $I \in \mathcal{I}$.
- (I3) For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.
- (IM) Whenever $I \in \mathcal{I}$ and $I \subseteq X \subseteq E$, the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

Sets of linkable sets not defining a matroid

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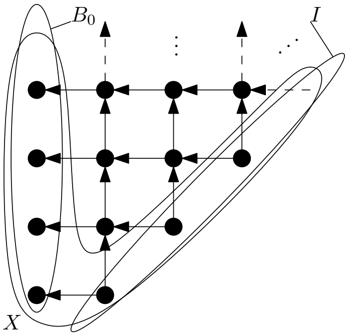
Half-grid



For $J \subseteq B_0$, $I \cup J$ is independent if and only if $B_0 \setminus J$ is infinite.

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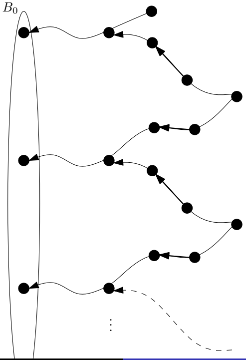
a set is maximally linkable if and only if it is linkable onto B_0.

An alternating ray is a ray with infinitely many in-degree 2 vertices. An alternating comb is obtained by identifying the first vertices of disjoint paths to B_0 with the in-degree 2 vertices of an alternating ray.



Alternating comb is the unique obstruction

Lemma (Afzali, L, Müller)

In (D, B_0), onto-linkability is equivalent to maximal linkability if and only if (D, B_0) contains no alternating comb.

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The Linkage Theorem (Pym 1969)

Let D be a digraph and two linkages be given: $\mathcal{P}$ from $X_\mathcal{P}$ onto $Y_\mathcal{P}$ and $\mathcal{Q}$, from $X_\mathcal{Q}$ onto $Y_\mathcal{Q}$. Then there is a set X^∞ satisfying $X_\mathcal{P} \subseteq X^\infty \subseteq X_\mathcal{P} \cup X_\mathcal{Q}$ which is linkable onto a set Y^∞ satisfying $Y_\mathcal{Q} \subseteq Y^\infty \subseteq Y_\mathcal{Q} \cup Y_\mathcal{P}$.

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$\exists$ strict gammoid that cannot be defined by any dimaze not containing any alternating comb.

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Minors of gammoids

The class of all gammoids is closed under deletion.
Let $M = M_L(D, B_0)$ be a matroid. Contracting a set X is easy if $X \subseteq B_0$:

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The set of linkable sets of a dimaze not containing any outgoing comb or linking fan is finitary.

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Linking fan: a subdivided infinite star whose leaves are in B_0
With B' as the ray of an outgoing comb, we get a negative answer to the above question.

Minors of gammoids: shifting

Lemma (Afzali, L, Müller)

Suppose a dimaze (D, B_0) not containing any outgoing comb defines a strict gammoid M. For any base B' of M, there exists a digraph D' such that $M = M_L(D', B')$.

Theorem (Afzali, L, Müller)

The class of gammoids defined by dimazes not containing any outgoing comb is minor-closed.

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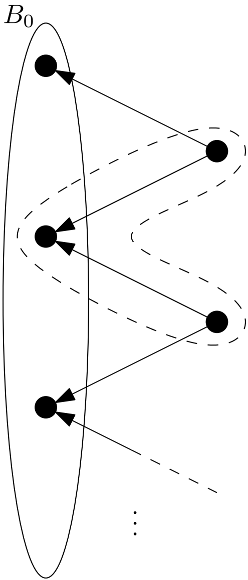
Duals of gammoids

Proposition (Afzali, L, Müller)

1. There is a strict gammoid that is not dual to any transversal matroid.
2. There is a transversal matroid that is not dual to any strict gammoid.
3. There is a strict gammoid, which is also a transversal matroid, that is not dual to any gammoid.

A non-cotransversal strict gammoid



A strict gammoid not dual to any gammoid

Sketch of a proof for 3.

Take a rooted tree where each vertex has infinitely many children. Take B_0 to be the vertices in alternate levels, starting from the root. Direct the edges towards B_0. Let M be the strict gammoid defined.

For $b \in B_0$, let C_b be the fundamental cocircuit of M with respect to B_0. Then for any undirected ray $b_1 x_1 b_2 x_2 \cdots$ from the root, $C := \bigcup_{k \in \mathbb{N}} C_{b_k} \setminus \{x_k : k \in \mathbb{N}\}$ is a cocircuit of M.

If M^* is a gammoid, to obtain a contradiction, we piece together linkages for $C_b - x$ together for a linkage for C. □

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Thank you!