Fragility, and excluded minors for the class of dyadic matroids

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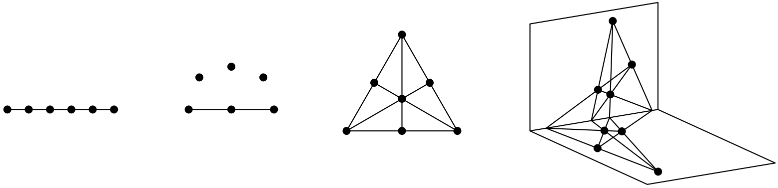
Joint work with Carolyn Chun, Deb Chun, Ben Clark, James Oxley, Charles Semple, Geoff Whittle, Alan Williams, Stefan van Zwam (and possibly others).

How to prove the GGK excluded-minor theorem

Theorem (Geelen, Gerards, Kapoor — 2000)

The excluded minors for the class of GF(4)-representable matroids are $U_{2,6}$, $U_{4,6}$, P_6, F_7^-, $(F_7^-)^*$, P_8, and P_8''.



$U_{2,6}$ P_6 F_7^- P_8

How to prove the GGK excluded-minor theorem

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{ccccc} e & f & g & h & i \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \alpha \\ 0 & \alpha & \alpha^2 & 1 & \alpha^2 \\ 1 & \alpha & 0 & 1 & 0 \end{bmatrix} \end{array} \longleftrightarrow \begin{array}{c} \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 & 0 & \alpha & \alpha^2 & 1 & \alpha^2 \\ 0 & 0 & 0 & 1 & 1 & \alpha & 0 & 1 & 0 \end{array}\right] \end{array}$$

We follow the convention that the groundset of a matroid labels the rows and columns of a representing matrix.

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We follow the convention that the groundset of a matroid labels the rows and columns of a representing matrix.

Deleting a column corresponds to deleting that element from the matroid. Deleting a row corresponds to contracting that element from the matroid. Pivoting swaps row and column labels.

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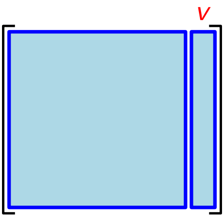
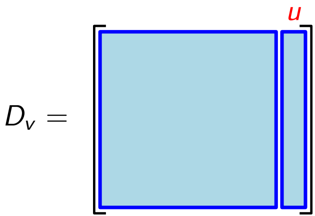
- Let M be an excluded minor for the class of GF(4)-representable matroids.
- Observe that M is 3-connected and contains a $U_{2,4}$-minor.
- Prove the existence of distinct elements u and v such that $M\backslash u$, $M\backslash v$, and $M\backslash u\backslash v$ are all 3-connected with $U_{2,4}$-minors.

How to prove the GGK excluded-minor theorem

$D_u =$ 

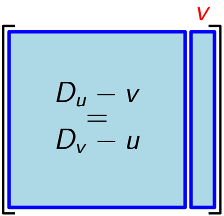
Since $M\backslash u$ is GF(4)-representable, we can represent it with a matrix D_u over GF(4), where v labels a column.

How to prove the GGK excluded-minor theorem



Similarly, $M \backslash v$ is represented by a matrix D_v over GF(4), where u labels a column.

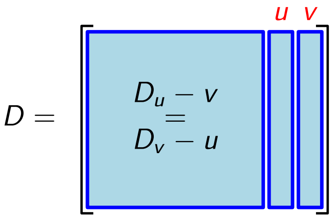
How to prove the GGK excluded-minor theorem

$D_v =$ 

Similarly, $M \backslash v$ is represented by a matrix D_v over GF(4), where u labels a column.

Both $D_u - v$ and $D_v - u$ represent the 3-connected matroid $M \backslash u \backslash v$. This matroid has a $U_{2,4}$-minor. Kahn proved that 3-connected matroids with $U_{2,4}$-minors have at most one representation over GF(4). Hence we can assume that $D_u - v = D_v - u$.

How to prove the GGK excluded-minor theorem



We adjoin the columns u and v to $D_u - v = D_v - u$. Call the resulting matrix D. Now D represents a matroid N over GF(4).

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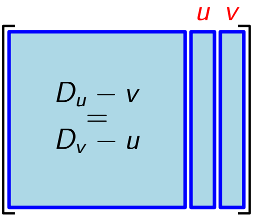
$D =$ 

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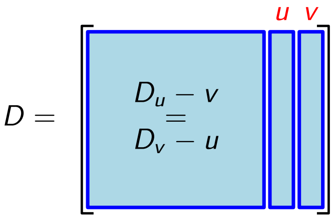

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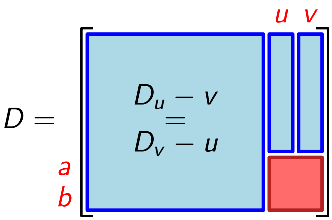

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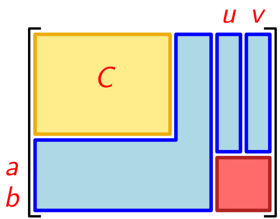
There is a subdeterminant that certifies $N \neq M$. This subdeterminant must use u and v. By pivoting, we can assume it is 2×2.

How to prove the GGK excluded-minor theorem



Let C be a minimal submatrix of $D - \{u, v\}$ such that C represents a matroid with a $U_{2,4}$-minor. Let M_C be this matroid.

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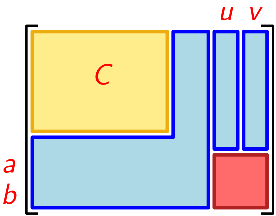

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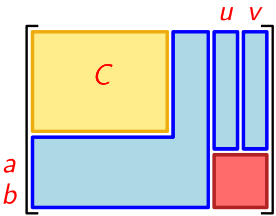
If e labels a row of C, then M_C/e has no $U_{2,4}$-minor. If e labels a column of C, then $M_C\backslash e$ has no $U_{2,4}$-minor. Therefore we say that M_C is $\{U_{2,4}\}$-fragile.

How to prove the GGK excluded-minor theorem



A GF(4)-representable $\{U_{2,4}\}$-fragile matroid must be a whirl (up to series and parallel elements).

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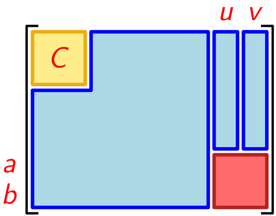

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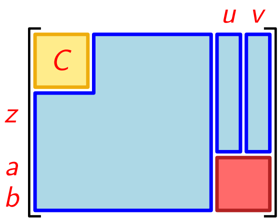

Now assume that there is a row, z, not in C or $\{a, b\}$.

Note N/z is a GF(4)-representable matroid, and deleting u or v from N/z produces a matroid identical to $M/z\backslash u$ or $M/z\backslash v$.

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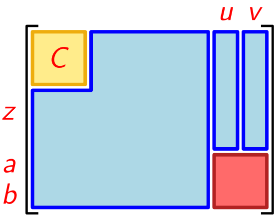

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We can make exactly the same statement about M/z.

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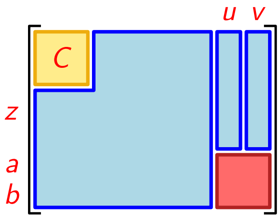

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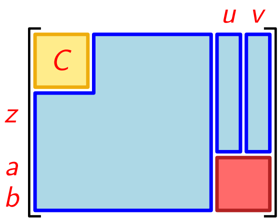

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Kahn's Theorem, and the fact that $M\backslash u\backslash v$ has a $U_{2,4}$-minor, implies that $N/z = M/z$.

However, the subdeterminant $D[\{a, b, u, v\}]$ implies $N/z \neq M/z$.

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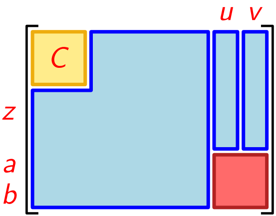


This contradiction shows that there are at most four rows.

We can similarly show that there are at most four columns.

Hence an excluded minor has at most eight elements. The remainder of the proof is case-checking.

How can we generalise these methods?

Definition

A partial field, $\mathbb{P} = (R, G)$, consists of R, a commutative ring with identity, and G, a subgroup of the group of units such that $-1 \in G$.

The aim is to characterise the excluded minors for representability over a partial field, $\mathbb{P}$. We proceed as before but instead of GF(4)-representable $\{U_{2,4}\}$-fragile matroids, we consider $\mathbb{P}$-representable $\mathcal{S}$-fragile matroids, where $\mathcal{S}$ is a set of strong stabilizers for $\mathbb{P}$-representable matroids.

Definition

M is $\mathcal{S}$-fragile if either $M \backslash e$ or M/e has no minor in $\mathcal{S}$, for every element e.

Key Point: Understanding $\mathbb{P}$-representable $\mathcal{S}$-fragile matroids helps to find the excluded minors for $\mathbb{P}$-representable matroids.

The fragility strategy

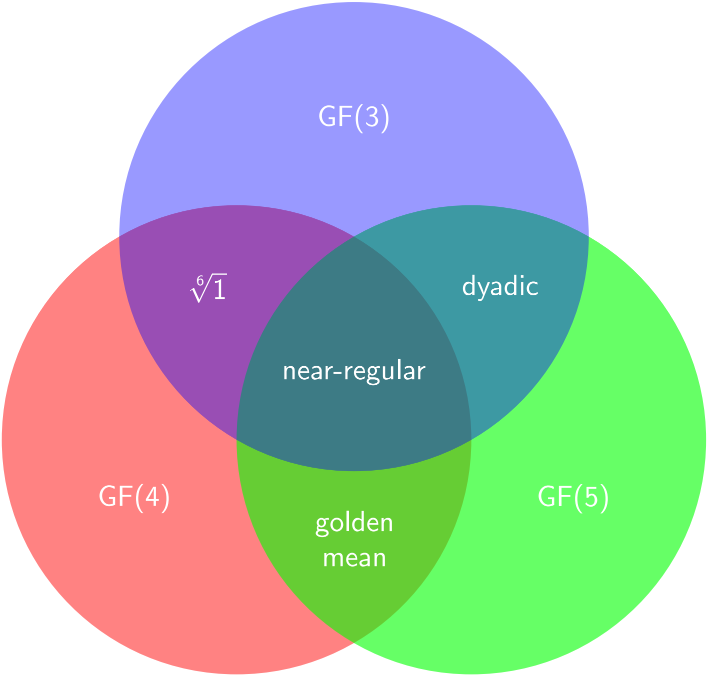
Definition

The near-regular partial field is $(\mathbb{Q}(\alpha), \{\pm\alpha^i(1-\alpha)^j : i, j \in \mathbb{Z}\})$.

The fragility strategy lets us characterise the excluded minors for near-regular matroids.

Theorem (Hall, Mayhew, Van Zwam — 2010)

The excluded minors for near-regular matroids are $U_{2,5}$, $U_{3,5}$, F_7, $(F_7)^*$, F_7^-, $(F_7^-)^*$, P_8, $\mathrm{AG}(2,3)\backslash e$, $(\mathrm{AG}(2,3)\backslash e)^*$, and $\Delta_T(\mathrm{AG}(2,3)\backslash e)$.


GF(3)
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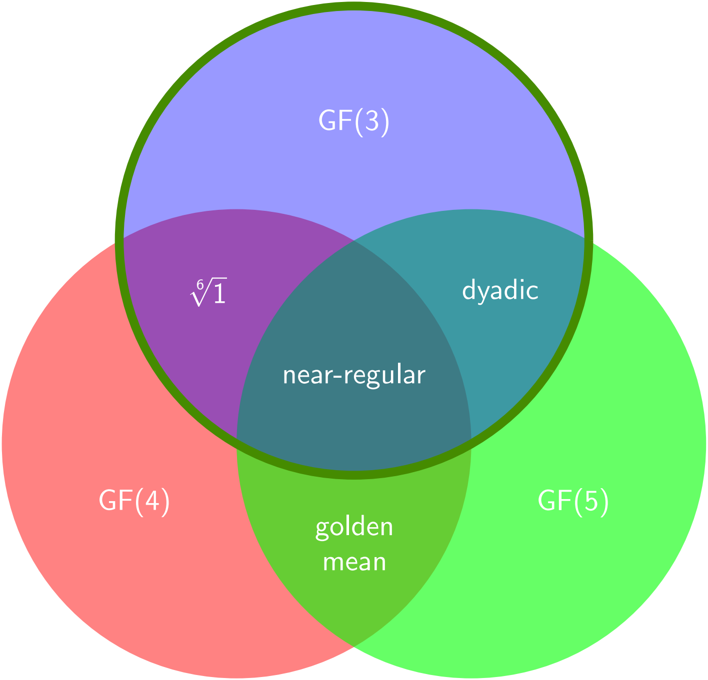
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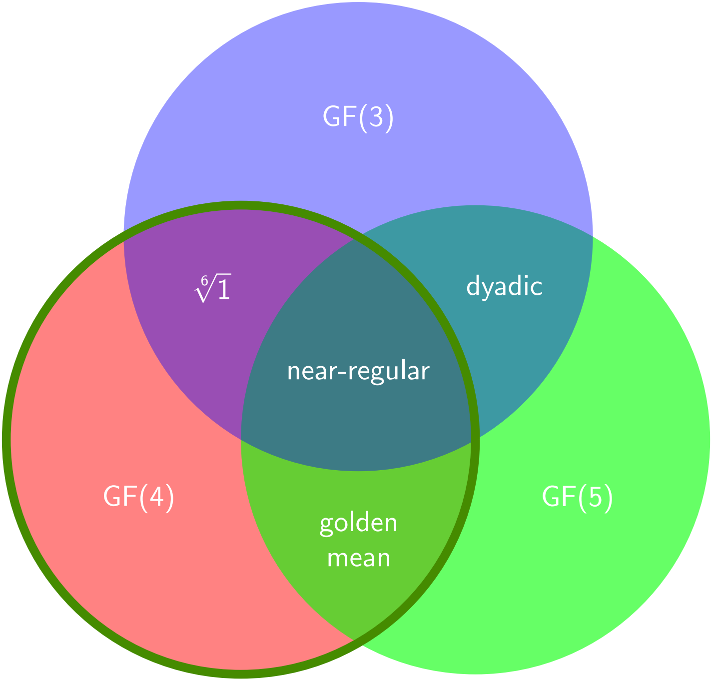
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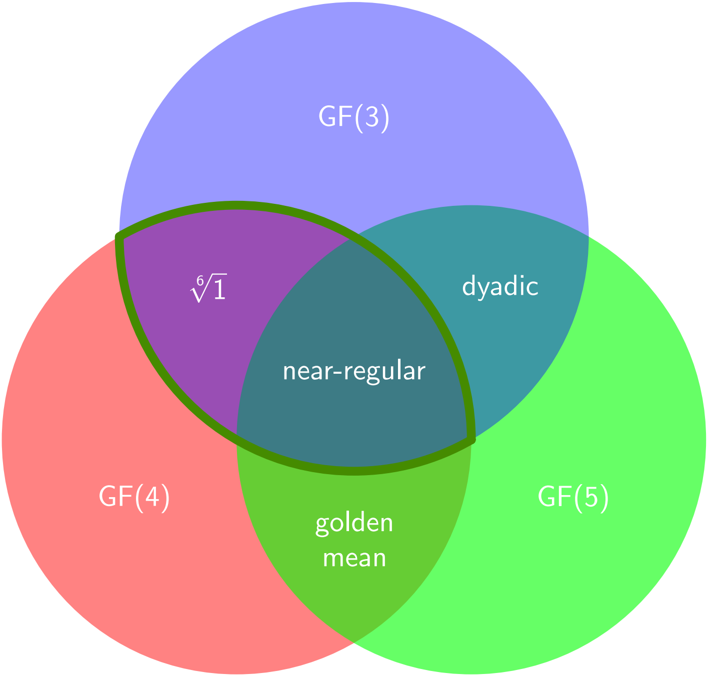
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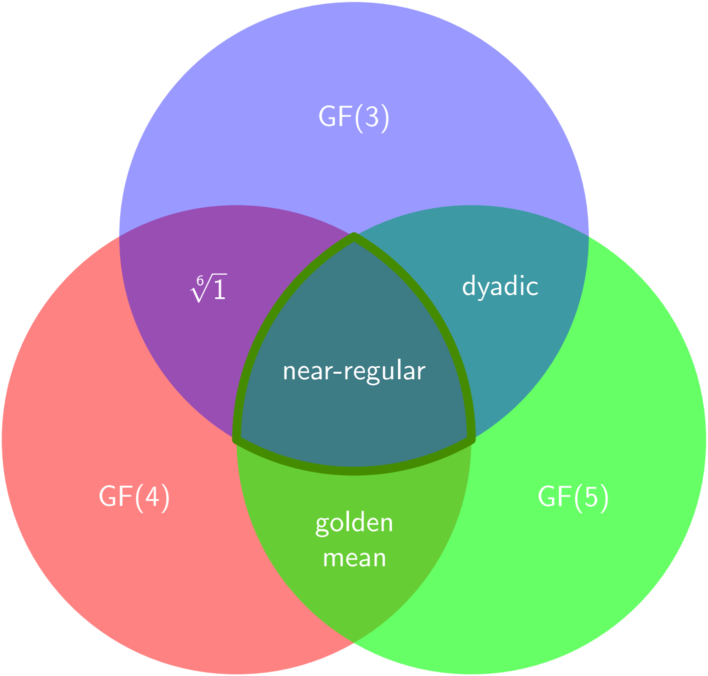

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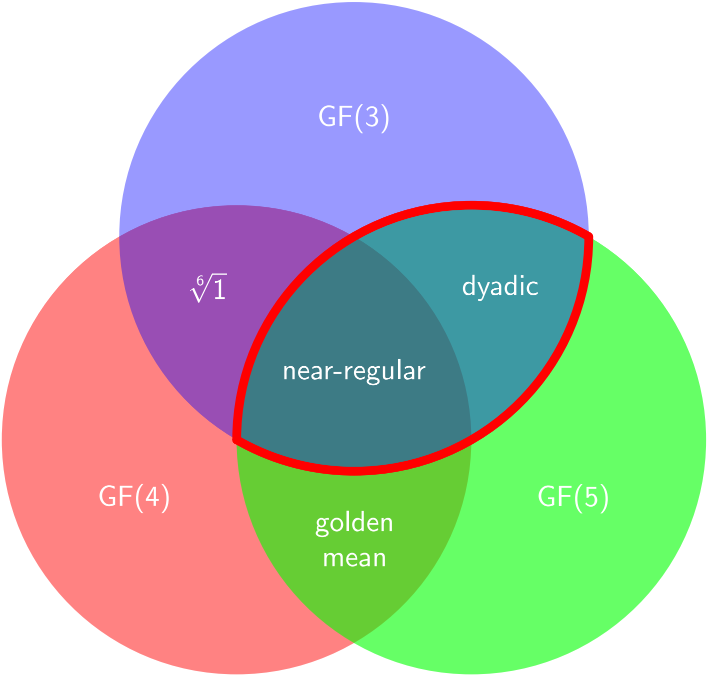
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Dyadic excluded minors

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A matroid is dyadic if and only if it is representable over GF(3) and GF(5).

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- Seven of those excluded minors are also excluded minors for dyadic matroids, so we can eliminate those cases. Therefore we assume that M properly contains F_7^-, $(F_7^-)^*$, or P_8.

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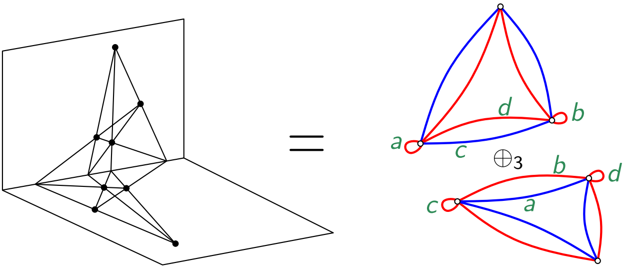
- Seven of those excluded minors are also excluded minors for dyadic matroids, so we can eliminate those cases. Therefore we assume that M properly contains F_7^-, $(F_7^-)^*$, or P_8.
- F_7^-, $(F_7^-)^*$, and P_8 are strong stabilizers for dyadic matroids, so we consider dyadic $\{F_7^-, (F_7^-)^*, P_8\}$-fragile matroids.

Dyadic $\{F_7^-, (F_7^-)^*, P_8\}$-fragile matroids

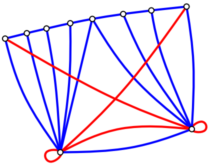
P_8 is the sum of two signed-graphic matroids.

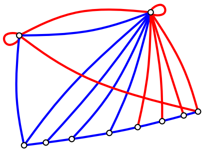


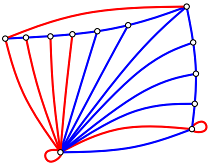
Dyadic $\{F_7^-, (F_7^-)^*, P_8\}$-fragile matroids

Work in progress (Mayhew, Whittle, Van Zwam)

Up to duality and a handful of sporadic matroids, a dyadic $\{F_7^-, (F_7^-)^*, P_8\}$-fragile matroid with a P_8-minor is the sum of signed-graphic matroids belonging to three possible families.







This brings us closer to the excluded minors for dyadic matroids.