Minimal Non-orientable Matroids of rank 3

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Joint work with
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- Hidefumi Hiraishi and Sonoko Moriyama, A new infinite family of minimal non-orientable matroids of rank 3 with 3n elements, Electronic Notes in Discrete Mathematics, vol. 44, pp. 275-280, 2013.
- Hidefumi Hiraishi and Sonoko Moriyama, Minimal non-orientable matroids of rank 3, submitted.

Matroids and Oriented Matroids

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$\det(a_1\ a_2\ a_3) = 1$ $\det(a_1\ a_3\ a_5) = -1$

$\det(a_1\ a_2\ a_4) = 0$ $\det(a_2\ a_3\ a_5) = 0$

$\det(a_1\ a_3\ a_4) = -1$ $\det(a_1\ a_4\ a_5) = 1$

$\det(a_2\ a_3\ a_4) = 1$ $\det(a_2\ a_4\ a_5) = -1$

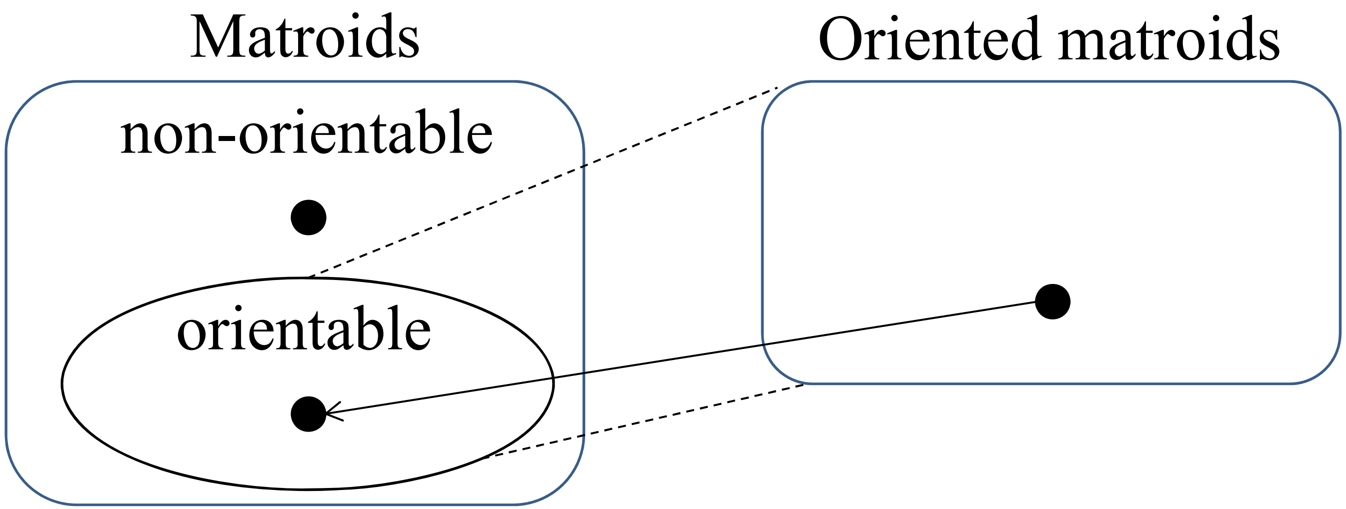
$\det(a_1\ a_2\ a_5) = 1$ $\det(a_3\ a_4\ a_5) = 1$

	1	1	1	2	1	1	2	1	2	3
	2	2	3	3	2	3	3	4	4	4
	3	4	4	4	5	5	5	5	5	5
Oriented Matroids	+	0	−	+	+	−	0	+	−	+
Matroids	*	0	*	*	*	*	0	*	*	*

Combinatorial abstraction of linear dependency

Orientability of matroids

- Every oriented matroid uniquely induces the (underlying) matroid
- There exist a matroid which is not realized as an oriented matroid



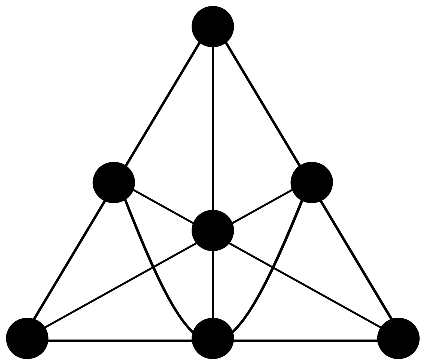
Definition:
If a matroid M is realized as an oriented matroid M', M is said to be orientable. M is the underlying matroid of M'.

… Characterization of the class of orientable matroids
by listing minimal non-orientable matroids …

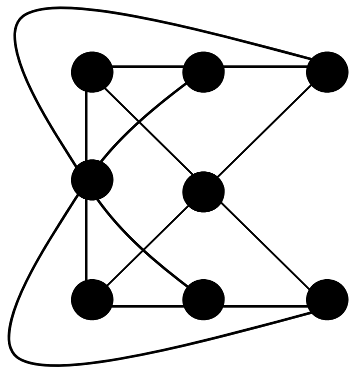
Minimal non-orientable matroids

A minimal non-orientable matroid
= an excluded minor of orientable matroids

The Fano matroid F_7



The MacLane matroid ML_8



- [Robertson, Seymour (2004)] Graph minor theorem:
  A family of graphs is minor-closed
  if and only if it has a finite number of excluded minors
- Orientable matroids are closed under minors
  ➔ Is #minimal non-orientable matroids finite or not?

History of minimal non-orientable matroids

[Bland, Las Vergnas (1978)] rank r , $2r$ elements

- The number of minimal non-orientable matroids is infinite

[Ziegler (1991)] rank 3 , $3n-1$ elements

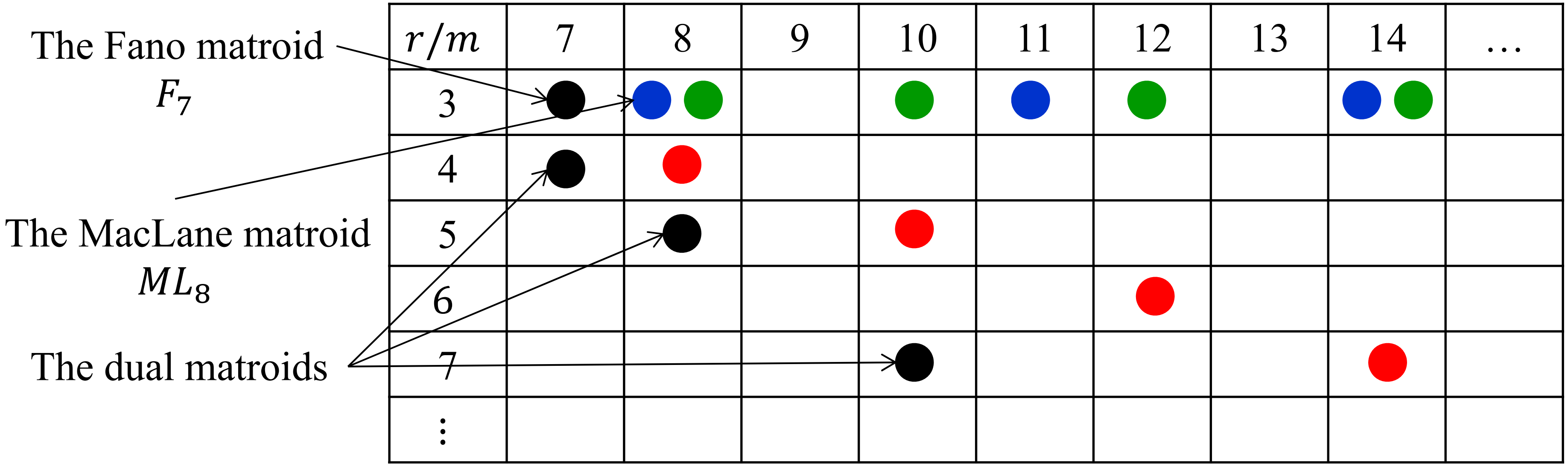
- Even in $r=3$, the number of minimal non-orientable matroids is infinite
- The infinite family starts with the MacLane matroid ML_8

[Flórez, Forge (2007)] rank 3 , $2n$ elements

- The infinite family starts with the MacLane matroid ML_8

[da Silva (2009)] rank r , $2r$ elements

- The infinite family is an extension of [Bland, Las Vergnas (1978)]



Our results

[Ziegler (1991)] rank 3 , $3n - 1$ elements

r/m	7	8	9	10	11	12	13	14	…
3	●	● ●	●	● ●	●	● ●	●	● ●	
4	●	●							
5		●		●					
6						●			
7				●				●	
⋮									

Is there a minimal non-orientable matroid of rank r on n elements?

Two infinite families of minimal non-orientable matroids

- rank 3, $3n - 2$ elements … starting with the Fano matroid F_7
- rank 3, $3n$ elements

Theorem: There exists a minimal non-orientable matroid of rank 3 on m elements for every $m \geq 7$.

Overview of this talk

- Two infinite families of matroids of rank 3
  - Extraction of minimal non-orientable matroids
  - Definition of two infinite families
- Proofs on minimal non-orientability
  - Review of the proof [Ziegler (1991)]
  - Proofs on non-orientability
  - Proofs on minimality

Enumeration of isomorph-free matroids

*: not completed

r/m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1		1											
2			1	1	1	1	1	1	1	1	1	1	1
3				1	2	4	9	23	68	383	5249	232928	28872972
4					1	3	11	49	617	185981	4884573865	*	*
5						1	4	22	217	188936	*	*	*
6							1	5	40	1092	4886374072	*	*
7								1	6	66	9742	*	*
8									1	7	104	298034	*
9										1	8	156	*
10											1	9	*
11											*	1	10
12											*	*	1

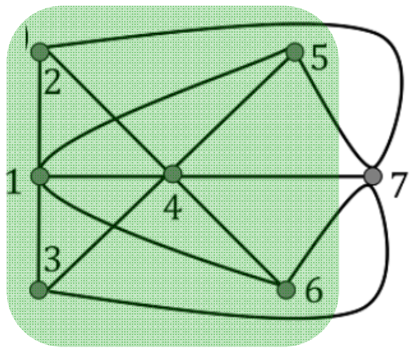
[Blackburn, Crapo, Higgs (1973)] [Betten, Betten (1999)] [Mayhew, Royle (2008)]
[Matsumoto, **M**, Imai, Bremner (2012)] … based on the enumeration algorithm of oriented matroids [Finschi, Fukuda (2001)]

http://www-imai.is.s.u-tokyo.ac.jp/~ymatsu/matroid/index.html

Two types of minimal non-orientable matroids

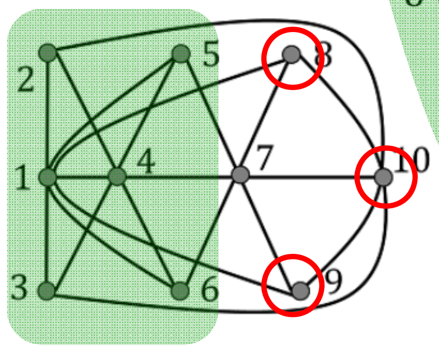
$r = 3$; m	7	8	9	10	11	12
# non-orientable	1	3	18	201	9413	1999921
# minimal non-orientable	1	1	2	23	1458	397240

Two types of minimal non-orientable matroids with similar structure

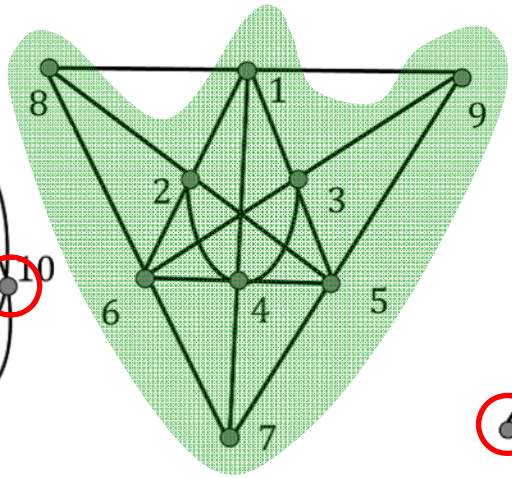


$m = 7$

The Fano matroid F_7

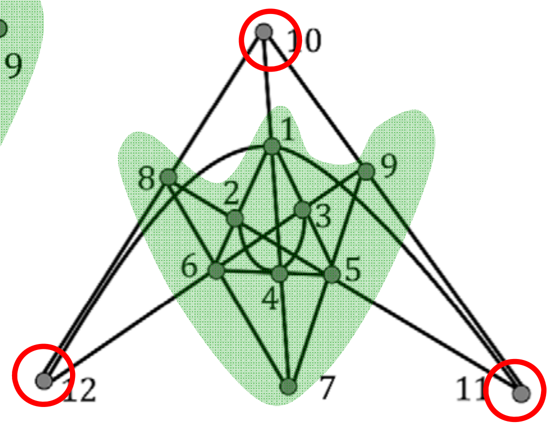


$m = 10$



$m = 9$

YM_9^1

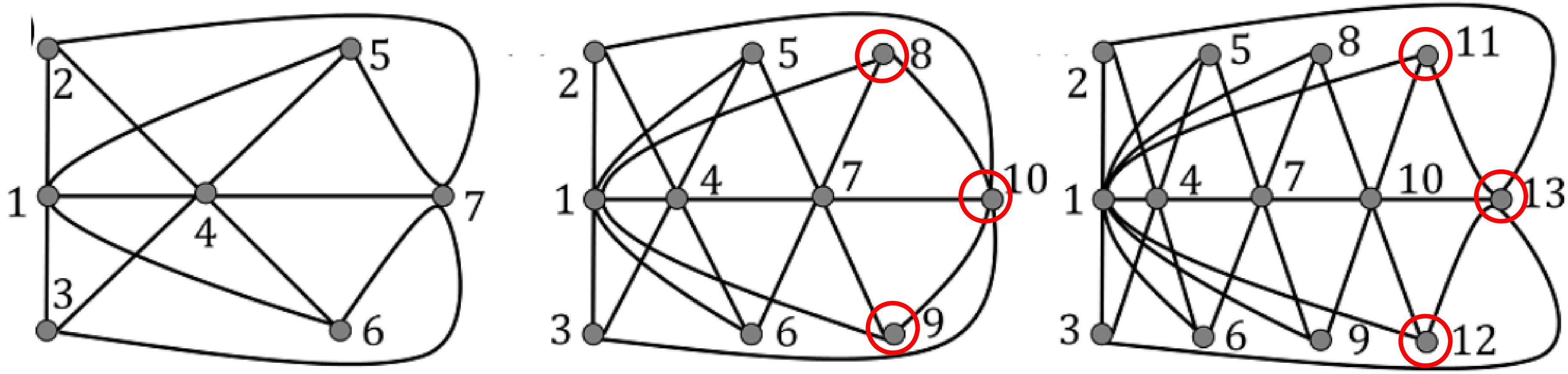


$m = 12$

The first infinite family of matroids F_{3n-2}

Definition: For every $n \geq 3$, F_{3n-2} is a simple matroid $(E_n^1, \mathcal{H}_n^1)$ where $E_n^1 := \{1, \cdots, 3n-2\}$, and $\mathcal{H}_n^1$ is the collection of the following hyperplanes:

- $\{1, 4, \cdots, 3n-2\}$; $\{1, 3k-1, 3k\}$ for $k = 1, 2, \cdots, n-1$
- $\{3k-1, 3k+1, 3k+3\}$ and $\{3k, 3k+1, 3k+2\}$
  for $k = 1, 2, \cdots, n-2$
- $\{a, 3n-4, 3n-2\}$ and $\{b, 3n-3\ 3n-2\}$
  where $a = 2, b = 3$ if n is odd, otherwise $a = 3, b = 2$
- all two-element subsets not contained in any above subsets

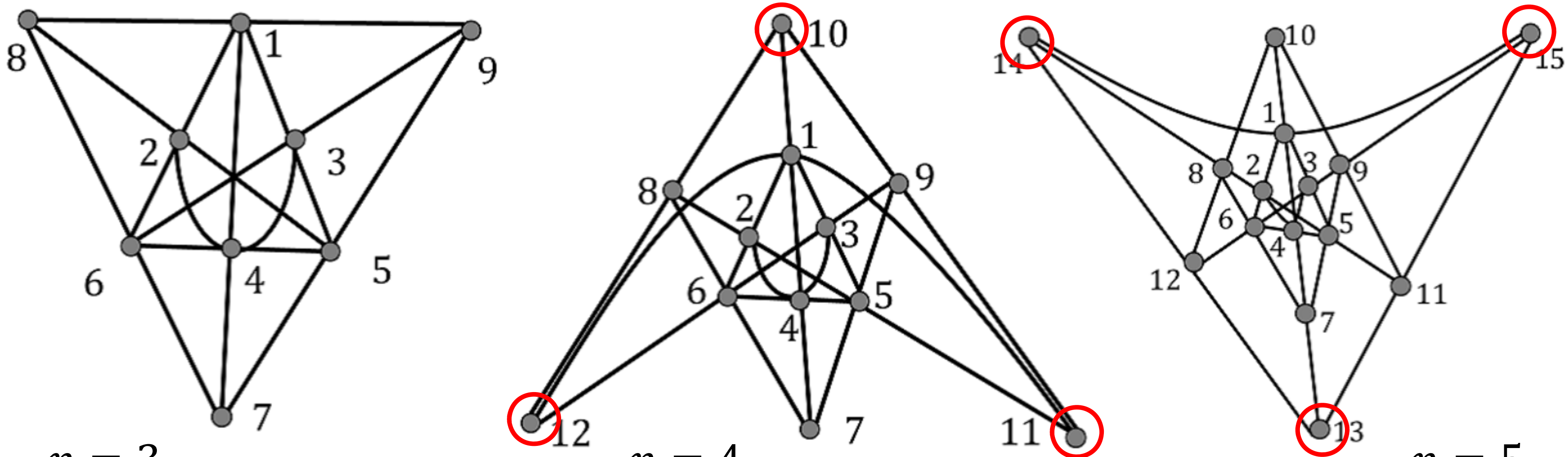


$n = 3$ $n = 4$ $n = 5$

The second infinite family of matroids YM^1_{3n}

Definition: For every $n \geq 3$, YM^1_{3n} is a simple matroid $(E^2_n, \mathcal{H}^2_n)$ where $E^2_n \coloneqq \{1, \cdots, 3n\}$, and $\mathcal{H}^2_n$ is the collection of the following hyperplanes:

- $\{1, 4, \cdots, 3n-2\}$, $\{2, 5, \cdots, 3n-1\}$ and $\{3, 6, \cdots, 3n\}$
- $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, $\{4, 5, 6\}$
- $\{1, 3n-1, 3n\}$; $\{3k, 3k+1, 3k+2\}$ and
  $\{3k-1, 3k+1, 3k+3\}$ for $k = 2, \cdots, n-1$
- all two-element subsets not contained in any above subsets



$n = 3$ $n = 4$ $n = 5$

Overview of this talk

- Two infinite families of matroids of rank 3
  - Extraction of minimal non-orientable matroids
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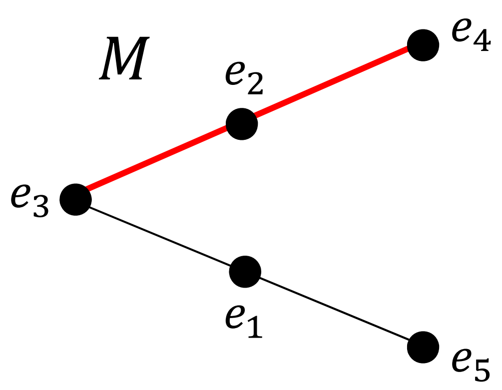
  - Proofs on non-orientability
  - Proofs on minimality

Properties on orientable matroids

An orientable matroid = the underlying matroid of an oriented matroid

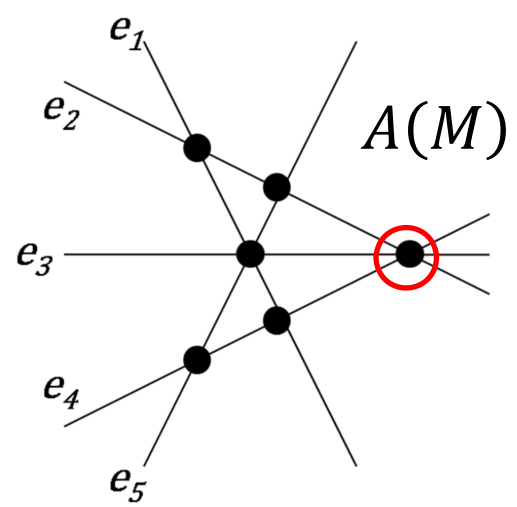
[Folkman, Lawrence (1978)] Topological Representation Theorem:
Every oriented matroid can be represented
by an arrangement of pseudohyperplanes

An orientable matroid of rank 3



A hyperplane of M
(= A flat of rank 2)

An arrangement of pseudolines

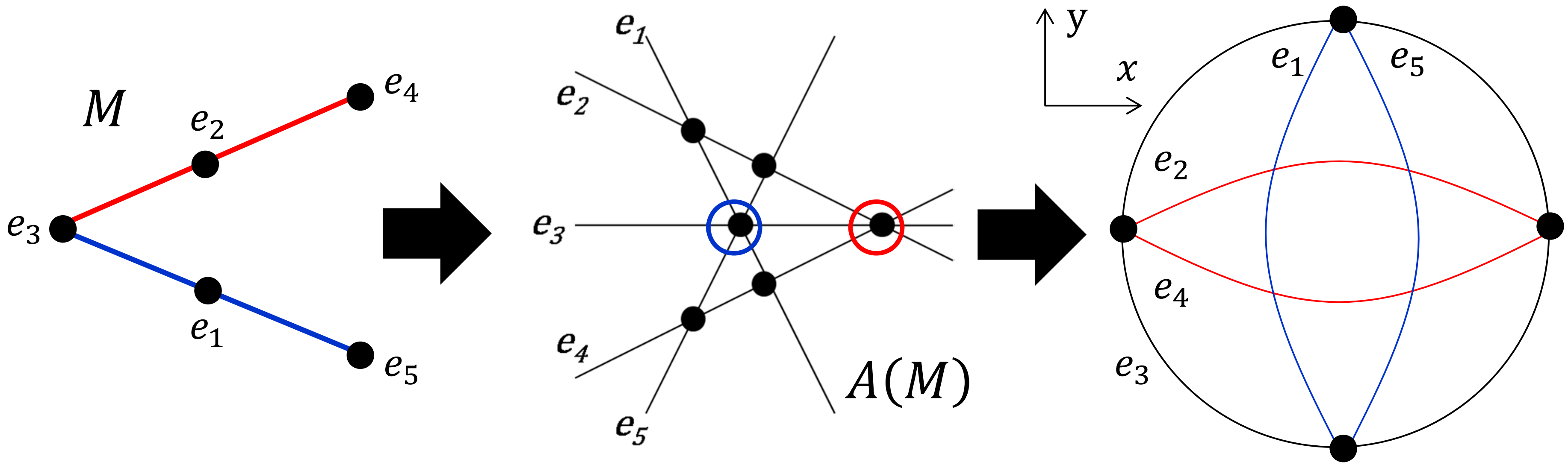


A maximal set of pseudolines
whose intersection is one point

[Ziegler (1991)] A key idea on orientable matroids

Theorem: Let $M := (E, \mathcal{H})$ be a matroid of rank 3 such that $\mathcal{H}$ includes two distinct hyperplanes H_1, H_2 whose intersection $H_1 \cap H_2$ consists of one element $e \in E$.
If M is orientable, an arrangement of pseudolines $A(M)$ includes the following grid $G(e, H_1 \setminus \{e\}, H_2 \setminus \{e\})$:

- the pseudoline e is at infinity
- the pseudolines $H_1 \setminus \{e\}$ are parallel to the x-axis, and
- the pseudolines $H_2 \setminus \{e\}$ are parallel to the y-axis

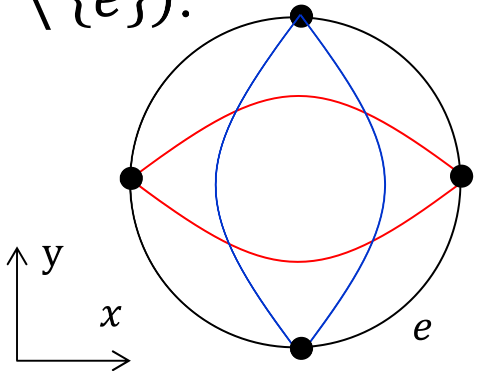


[Ziegler (1991)] Sketch of the proof for non-orientability

A matroid $M := (E, \mathcal{H})$ of rank 3 such that $\mathcal{H}$ includes two distinct hyperplanes H_1, H_2 whose intersection consists of one element $e \in E$

If M were orientable, an arrangement of pseudolines $A(M)$ would include the following grid $G(e, H_1 \setminus \{e\}, H_2 \setminus \{e\})$:

- the pseudoline e is at infinity
- the pseudolines $H_1 \setminus \{e\}$ are parallel to the x-axis, and
- the pseudolines $H_2 \setminus \{e\}$ are parallel to the y-axis



The other pseudolines $E \setminus \{H_1 \cup H_2\}$ cannot be added to the grid $G(e, H_1 \setminus \{e\}, H_2 \setminus \{e\})$ satisfying the hyperplanes $\mathcal{H}$

M is determined to be non-orientable

The proof for non-orientability of F_{3n-2}

Theorem: For every $n \geq 3$, F_{3n-2} is non-orientable.

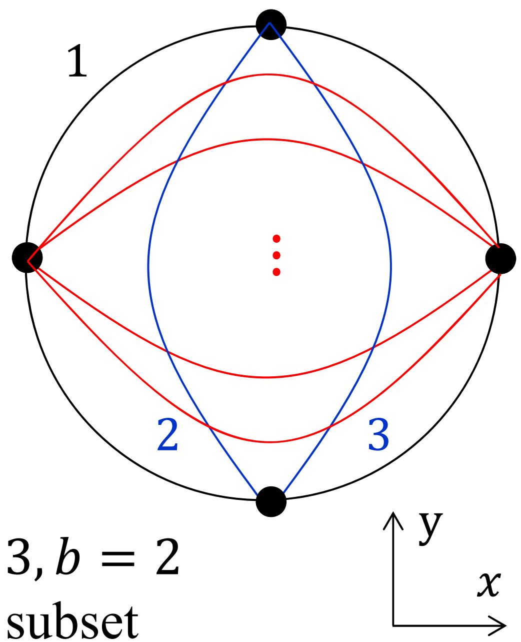
PROOF:

- $F_{3n-2} \coloneqq (E_n^1, \mathcal{H}_n^1)$ includes two distinct hyperplanes H_1, H_2 whose intersection consists of one element $1 \in E_n^1$

  $E_n^1 \coloneqq \{1, \cdots, 3n-2\}$

  $\mathcal{H}_n^1$ is the family of the following subsets of E_n^1:

  - $\{1, 4, \cdots, 3n-2\} (\coloneqq H_1)$
  - $\{1, 2, 3\} (\coloneqq H_2)$
  - $\{1, 3k-1, 3k\}$ for $k = 2, \cdots, n-1$
  - $\{3k-1, 3k+1, 3k+3\}$ and $\{3k, 3k+1, 3k+2\}$ for $k = 1, 2, \cdots, n-2$
  - $\{a, 3n-4, 3n-2\}$ and $\{b, 3n-3\ 3n-2\}$ where $a = 2, b = 3$ if n is odd, otherwise $a = 3, b = 2$
  - all two-element subsets not contained in any above subset



- Prove that the other pseudolines $E_n^1 \setminus \{H_1 \cup H_2\}$ cannot be added to the grid $G(1, H_1 \setminus \{1\}, H_2 \setminus \{1\})$ satisfying $\mathcal{H}_n^1$ [END]

The proof for non-orientability of YM_{3n}^1 (1)

Theorem: For every $n \geq 3$, YM_{3n}^1 is non-orientable.

PROOF:

Definition: For a matroid M on E, if an element $p \in E$ is contained in at most two hyperplanes of size at least three, p is reducible.

Theorem (Levi enlargement lemma): For a matroid M of rank 3 on E, let $p \in E$ be a reducible element. Then M is orientable if and only if $M \setminus \{p\}$ is orientable.

■ For every $n \geq 3$, construct the matroid ${YM'}_{3n+1}^1$ by adding one reducible element 0 to YM_{3n}^1

➔ ${YM'}_{3n+1}^1$ is orientable if and only if $YM_{3n}^1 \coloneqq {YM'}_{3n+1}^1 \setminus \{0\}$ is orientable

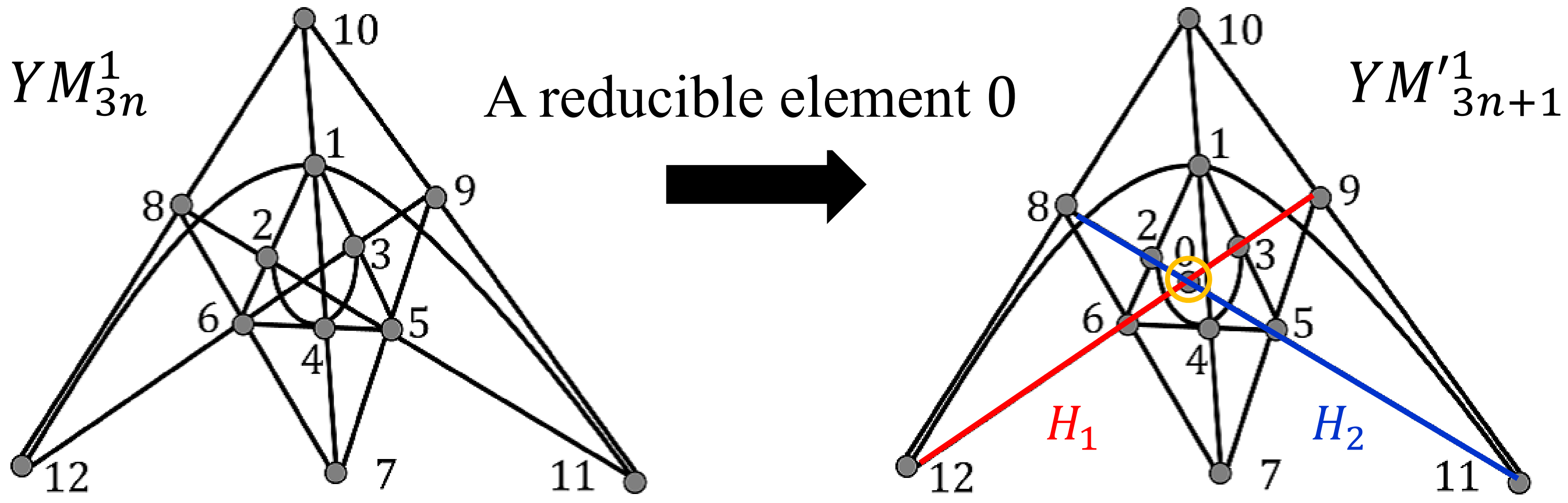
Prove that ${YM'}_{3n+1}^1$ is non-orientable ➔ YM_{3n}^1 is non-orientable

The proof for non-orientability of YM^1_{3n} (2)

Theorem: For every $n \geq 3$, YM^1_{3n} is non-orientable.

PROOF:

- $YM'^1_{3n+1} \coloneqq (E^3_n, \mathcal{H}^3_n)$ includes two distinct hyperplanes H_1, H_2 whose intersection consists of one element $0 \in E^3_n$



- Prove that there exist no arrangements of pseudolines including the grid $G(0, H_1 \setminus \{0\}, H_2 \setminus \{0\})$ satisfying $\mathcal{H}^3_n$
  ➔ YM'^1_{3n+1} is non-orientable ➔ YM^1_{3n} is non-orientable [END]

The proof for minimality of F_{3n-2} and YM_{3n}^1

Theorem: For every $n \geq 3$,
any proper minor of F_{3n-2} and YM_{3n}^1 is orientable.

PROOF:

- M: F_{3n-2} or YM_{3n}^1
- Contraction:
  The rank of any contraction of M is equal to two
  ➔ any contraction of M is orientable
- Deletion:
  - Prove that the deletion $M \setminus \{e\}$ is orientable for any e
  - Remove a reducible element from $M \setminus \{e\}$ one by one
    $M \setminus \{e\}$ ➔ $M \setminus \{e \cup e_1\}(=: M_1)$ (e_1: a reducible element of M) ➔...
    ➔ $M \setminus \{e \cup e_1 \cup \cdots \cup e_m\}$ $(=: M_m)$ (e_m: a reducible element of M_{m-1})
  - When the number of elements in M_m is less than 7,
    M_m is orientable ➔ …➔ $M \setminus \{e\}$ is orientable [END]

Concluding remarks

[Bland, Las Vergnas (1978)] [Ziegler (1991)]
[Flórez, Forge (2007)] [da Silva (2010)]

r/n	7	8	9	10	11	12	13	14	…
3	●	● ●	●	● ●	●	● ●	●	● ●	
4	●	●							
5		●		●					
6			●			●			
⋮									

rank r on $n = r + 3$ elements

Two infinite families of minimal non-orientable matroids

- The matroids F_{3n-2} of rank 3 on $3n - 2$ elements
- The matroids YM^1_{3n} of rank 3 on $3n$ elements

Theorem: There exists a minimal non-orientable matroid of rank 3 on n elements for every $n \geq 7$.

✓ Minimal non-orientable matroids of rank $r(\geq 4)$
✓ Relation between orientability and representability