Unavoidable connected matroids retaining a specified minor

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$M(G)$ has a k-element circuit or a k-element cocircuit as a minor.

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Let M be a 2-connected matroid whose largest circuit and largest cocircuit have c and c^* elements, respectively. Then

$$|E(M)| < 2^{c+c^*}.$$

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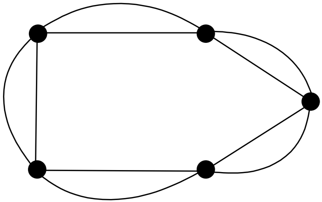

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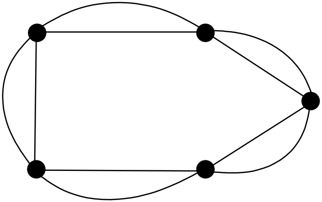

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Pou-Lin Wu (2000) determined all graphic matroids attaining the bound.

Problem (Royle)

Find all of the non-graphic matroids attaining the bound.

Capturing a particular element

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Let M be a 2-connected matroid whose largest circuit has c elements. Then every element of M is in a circuit with at least $\frac{c}{2} + 1$ elements.

Corollary

If e is an element of a sufficiently large 2-connected matroid M, then M has a big circuit or a big cocircuit containing e.

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Problem

Can we eliminate the elements that are not in either N or the big 2-connected uniform minor?

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Given a large 2-connected matroid M and a fixed 2-connected minor N of M, we want a 2-connected minor of M that has

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Not quite intertwining.

Examples

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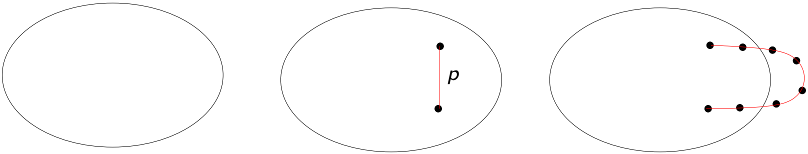
Extend or coextend N by a single element p to get N' and then take the 2-sum across p of N' and a big circuit containing p.

Equivalently, replace p by a large series class.



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- $M \backslash E(N)$ is 2-connected and uniform.

One of these two situations must arise.

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- *$E(M') - E(N)$ consists of a set of at least k clones; so*

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- *$M' \backslash E(N)$ is 2-connected and uniform with at least k elements.*

(ii) *N has a 2-conn single-element extension or coextension N' by p, such that M has, as a minor, the 2-sum across p of*

- *N'; and*
- *a circuit with at least k elements including p.*

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Lemma (Geelen, O, Vertigan, Whittle 1998)

Two elements are clones in a binary matroid if and only if

- *both are loops;*
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Corollary

Let N be a 2-conn binary matroid and k be a positive integer. Every sufficiently large 2-conn binary matroid M with N as a minor has a minor that is obtained by extending or coextending N by p and then adding $k-1$ elements in series or parallel with p.

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For a positive integer n, there is a positive integer $f(n)$ such that if M is 2-conn having both a circuit and a cocircuit with at least $f(n)$ elements, then M has one of the following as a minor:

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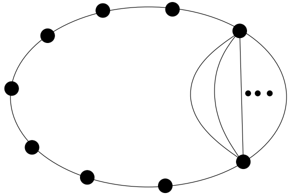

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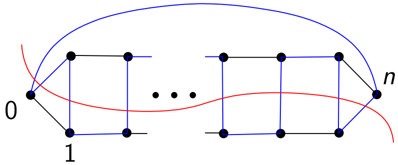
Graphic Examples

$\Gamma_1(n)$



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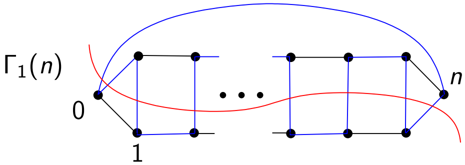
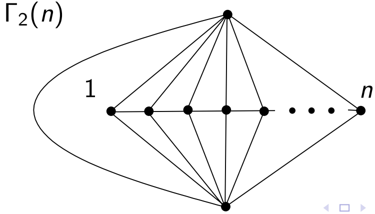
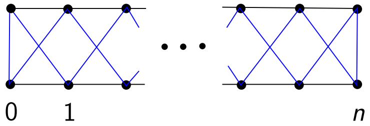




Big circuit-cocircuit intersection

More Graphic Examples

$\Gamma_3(n)$



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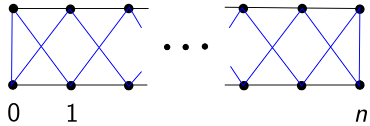

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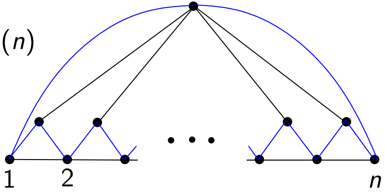

Big circuit-cocircuit intersection in graphs

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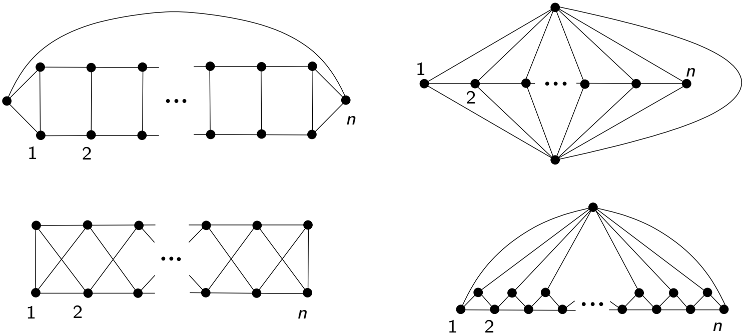
Theorem

For each $n \geq 3$, there is an $h(n)$ such that a graph with a set of size at least $h(n)$ that is the intersection of a cycle and a bond has one of the four graphs shown as a minor.



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Conjecture *For each $n \geq 3$, there is a $g(n)$ such that a binary matroid with a set of size at least $g(n)$ that is the intersection of a circuit and a cocircuit has, as a minor, $M[I_n | J_n - I_n]$ or the cycle matroid of one of the graphs $\Gamma_1(n), \Gamma_2(n), \Gamma_3(n)$, or $\Gamma_4(n)$.*

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Only progress: any matroid that needs to be added to the conjectured list of five unavoidable minors must be 3-connected.

Summary

A sufficiently large 2-conn matroid M with a fixed 2-conn matroid N as a minor has, up to duality, either

- a minor M' that has N as a spanning restriction such that $M' \backslash E(N)$ is a big 2-connected uniform matroid; or
- the 2-sum of a big circuit and a 2-conn single-element extension or coextension of N.

Summary

For $n \geq 3$, there is an $h(n)$ such that a graph with a cycle and bond having at least n common edges has one of the following four graphs as a minor.