Biased graphs with no vertex-disjoint unbalanced cycles

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Princeton
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(Joint work with R. Chen)

Theorem (Lovász)

Graphs with no two vertex-disjoint cycles:

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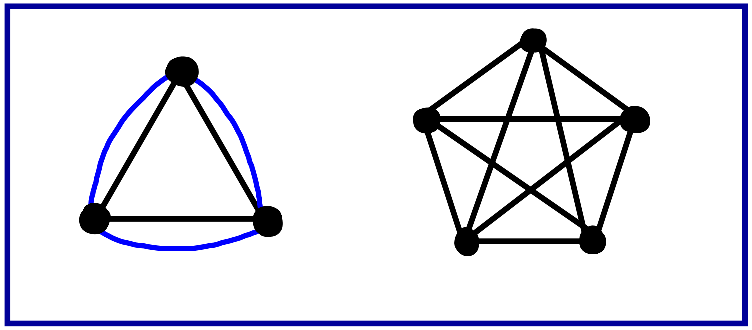

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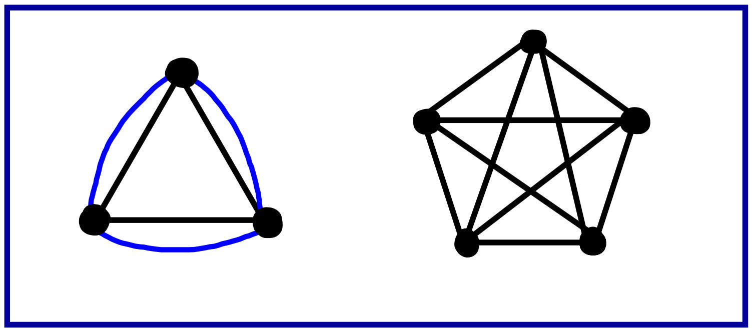

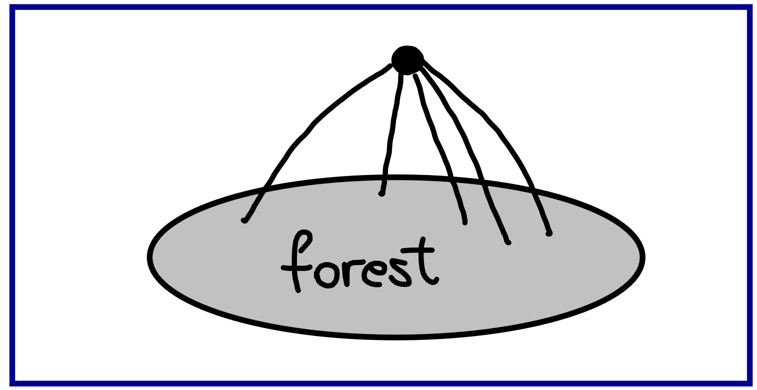

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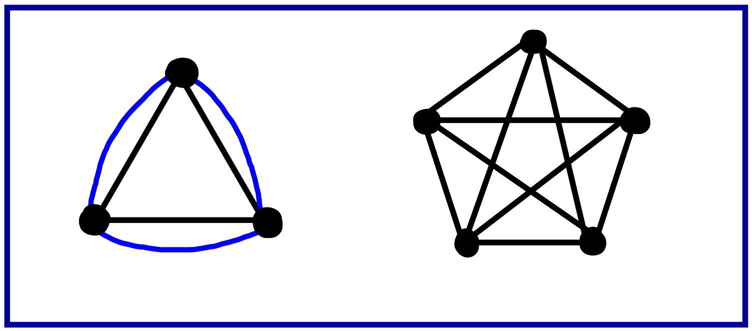

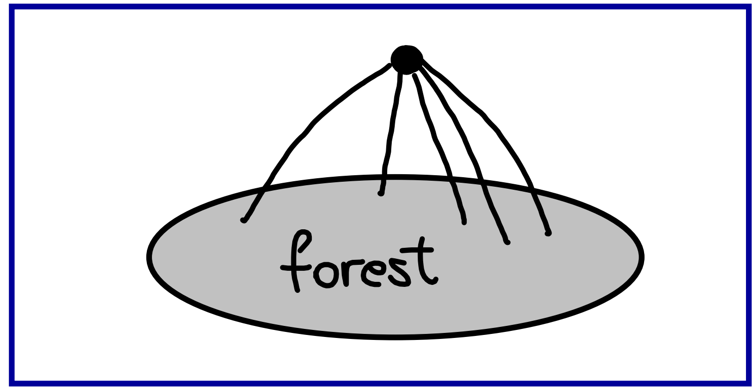

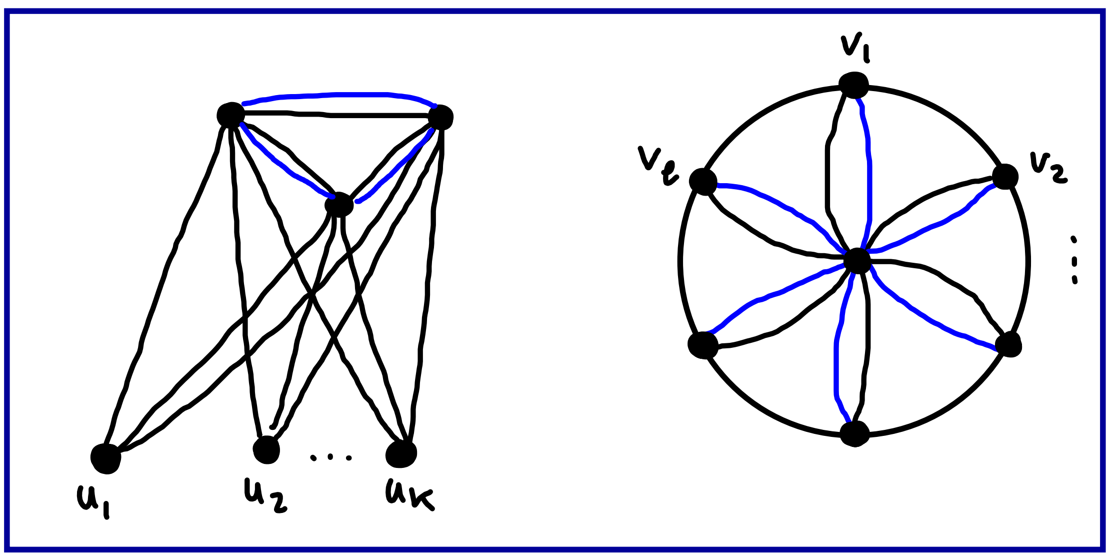

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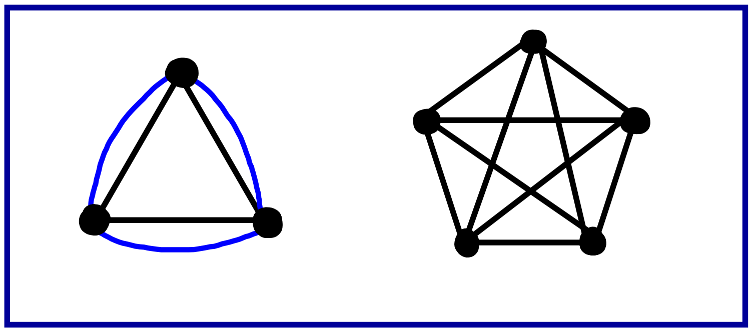
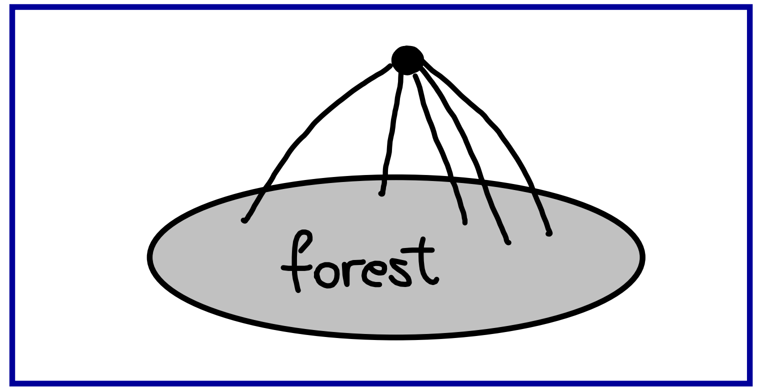
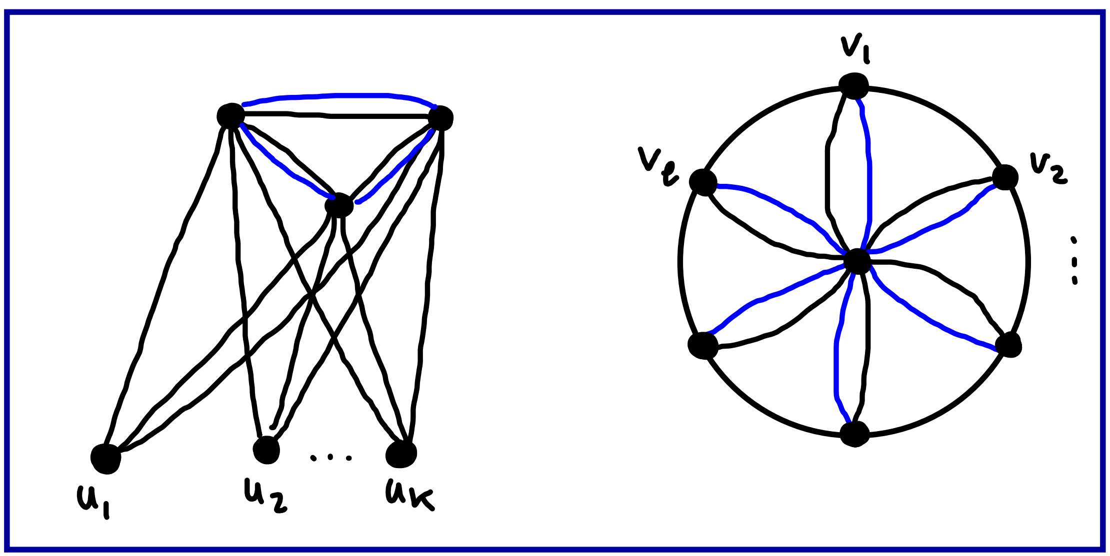


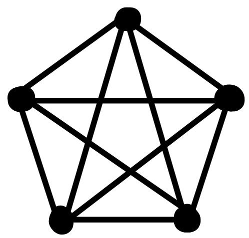


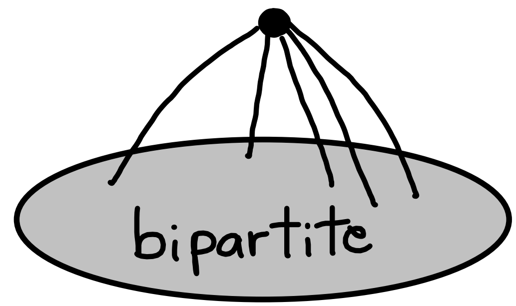


$\oplus_1$ trees

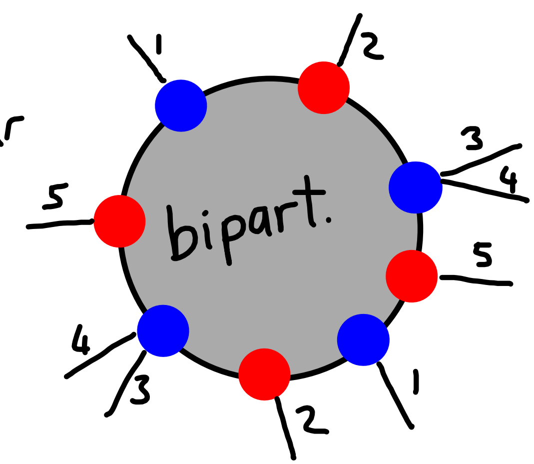
Theorem (Slilaty)

Graphs with no two vertex-disjoint odd cycles:





projective-planar with even face boundaries



$\oplus_1$, $\oplus_2$ bipartite pieces

Theorem (Slilaty)

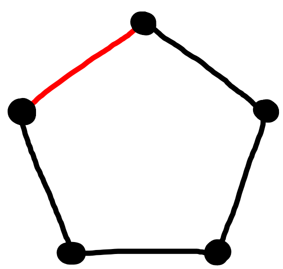
Signed graphs with no two vertex-disjoint S-odd cycles:

Theorem (Slilaty)

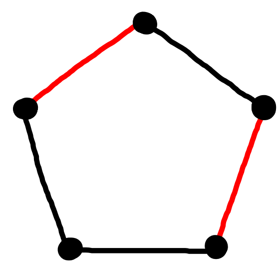
Signed graphs with no two vertex-disjoint S-odd cycles:

Signed graph: pair (G,S), where $S \subseteq E(G)$.

A cycle is S-odd if $|C \cap S|$ is odd



S-odd

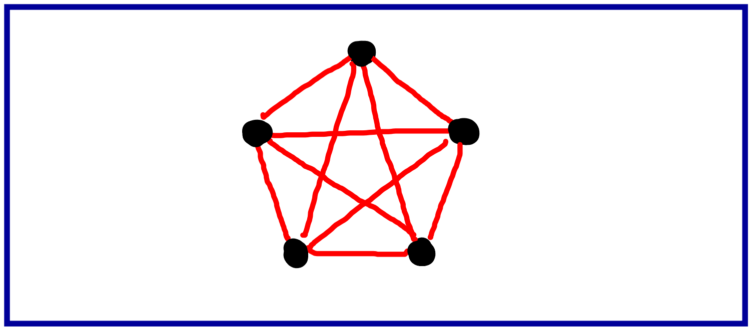


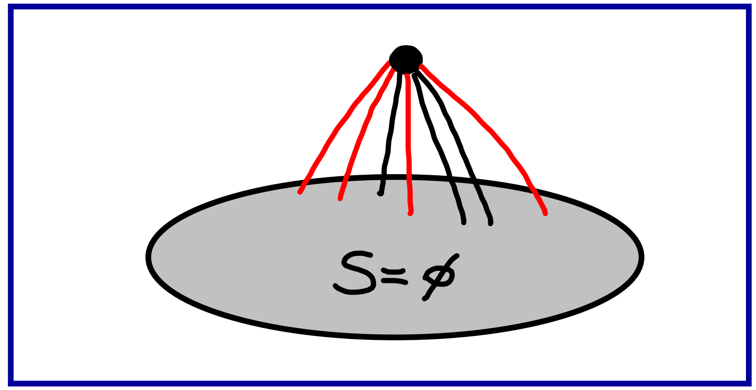
S-even

S

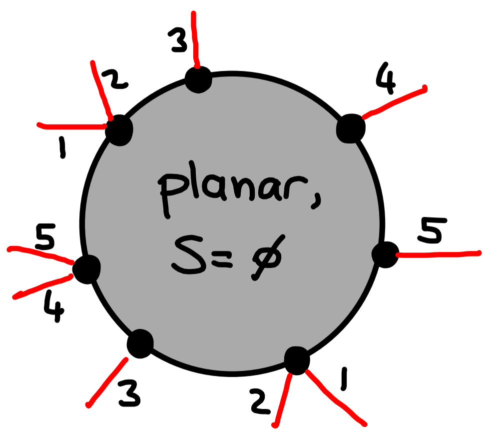
Theorem (Slilaty)

Signed graphs with no two vertex-disjoint S-odd cycles:



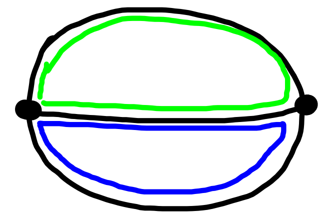


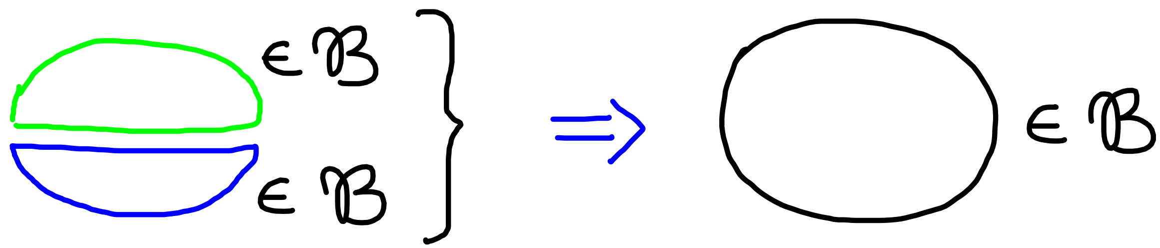
projective-planar signed graphic



$\oplus_1$, $\oplus_2$, $\oplus_3$ pieces with $S = \emptyset$

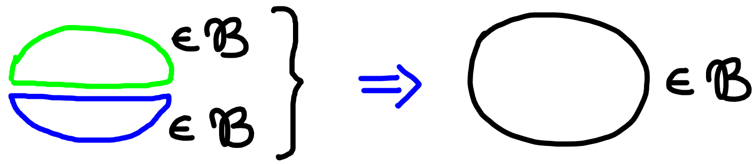
Biased graph: pair $(G, \mathcal{B})$ where G is a graph and $\mathcal{B}$ is a set of cycles of G satisfying the **theta property**:





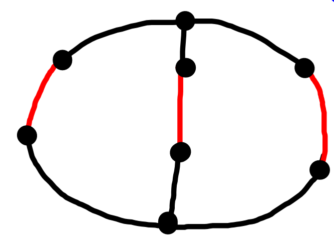
Cycles in $\mathcal{B}$ are **balanced**, cycles not in $\mathcal{B}$ are **unbalanced**.

Biased graph: pair $(G, \mathcal{B})$ where G is a graph and $\mathcal{B}$ is a set of cycles of G satisfying the theta property.

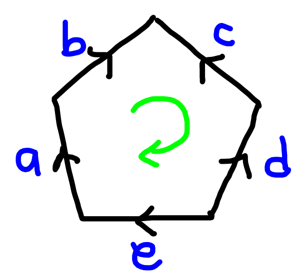


Examples:

- G with $\mathcal{B} = \emptyset$
- $(G, \mathcal{B})$ with $\mathcal{B} =$ all cycles of G (balanced biased graph)
- Signed graphs: $\mathcal{B} = \{\Sigma\text{-even cycles}\}$



- Group-labelled graphs:



$\in \mathcal{B}$ iff $abc^{-1}d^{-1}e = 1$

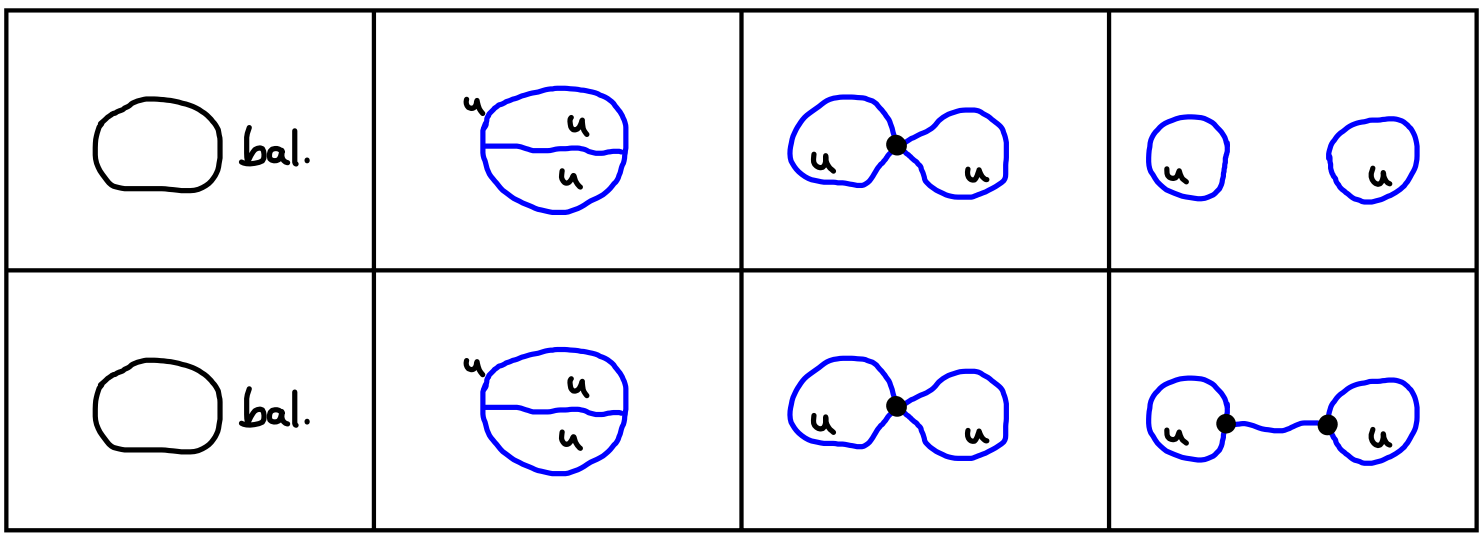
Biased graph $(G, \mathcal{B})$ → lift matroids

→ frame matroids

CIRCUITS:

lift

frame



Theorem (Chen, Pivotto)

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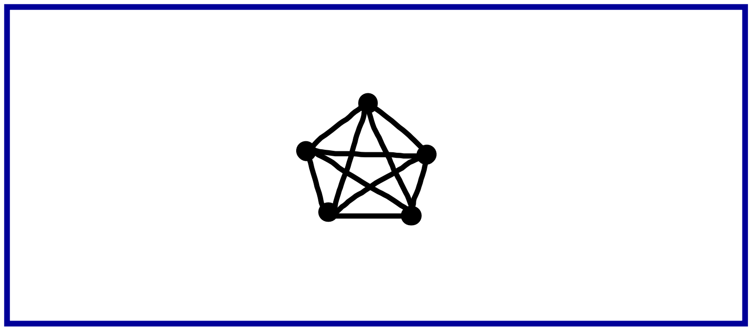

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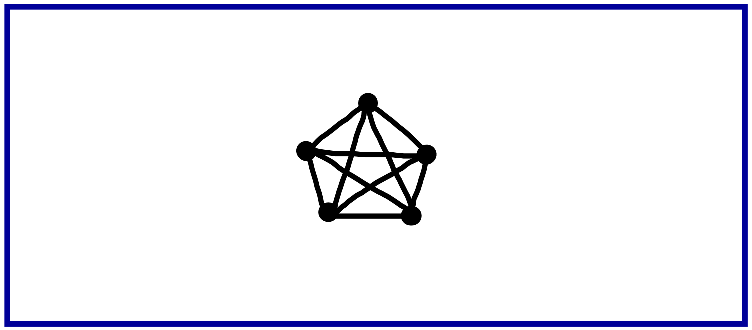

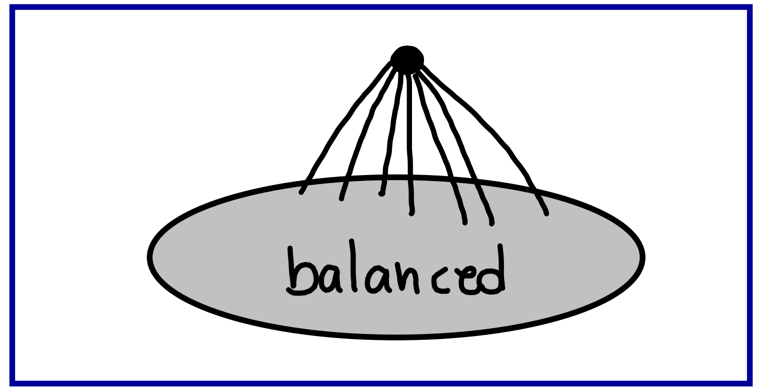

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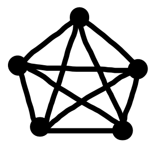

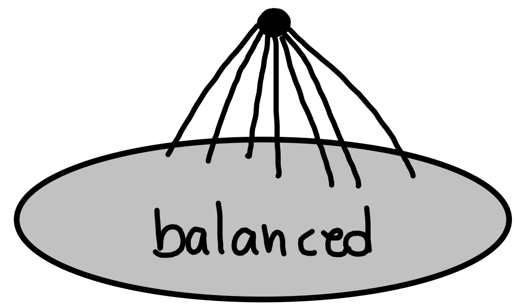

$\oplus_1$, $\oplus_2$, $\oplus_3$

balanced pieces

Theorem (Chen, Pivotto)

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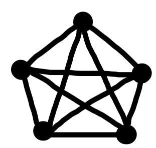

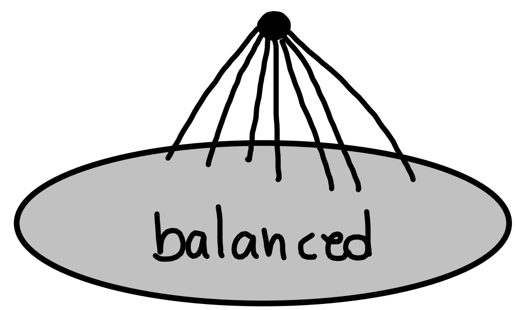

$\oplus_1$, $\oplus_2$, $\oplus_3$ balanced pieces

Proof idea: $\Omega = (G, \mathcal{B})$ biased graph with no two disjoint unbalanced cycles.

Step 0: exclude simple cases

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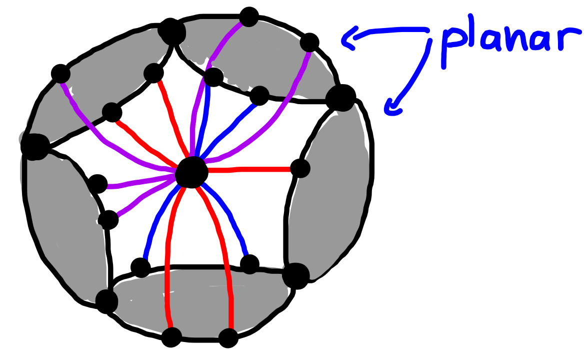
Step 0: exclude simple cases

Step 1: either G is 4-connected, or



Proof idea: $\Omega = (G, \mathcal{B})$ biased graph with no two disjoint unbalanced cycles.

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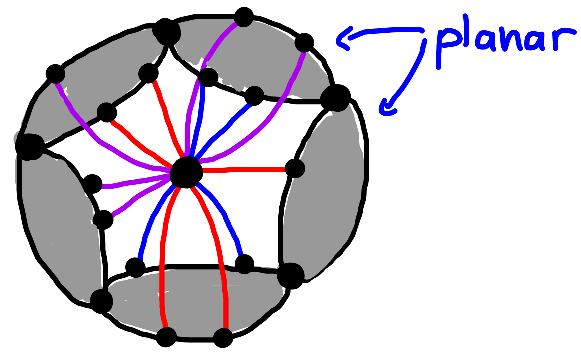
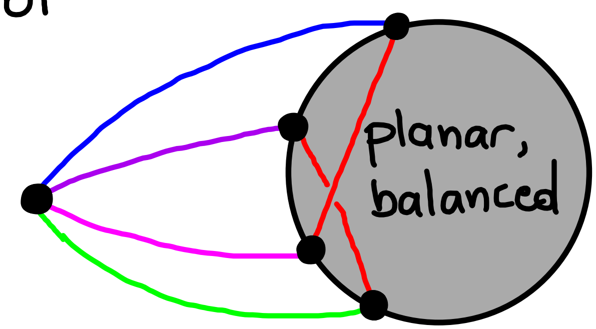


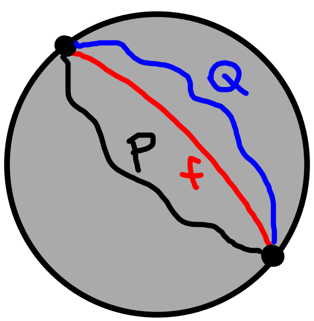
Step 2: either Ω has a 2-connected spanning balanced subgraph, or



Proof idea: $\Omega = (G, \mathcal{B})$ biased graph with no two disjoint unbalanced cycles.

Steps 0, 1, 2: G is 4-connected and Ω has a 2-connected spanning balanced subgraph Ω'. Pick Ω' maximal.

Set $U = E(\Omega) - E(\Omega')$.

$\Omega' \rightarrow$ 

$f \in U$

$\Rightarrow$ every cycle in $\Omega' \cup f$ using f is unbalanced

Proof idea: $\Omega = (G, \mathcal{B})$ biased graph with no two disjoint unbalanced cycles.

Steps 0, 1, 2: G is 4-connected and Ω has a 2-connected spanning balanced subgraph Ω'. Pick Ω' maximal.

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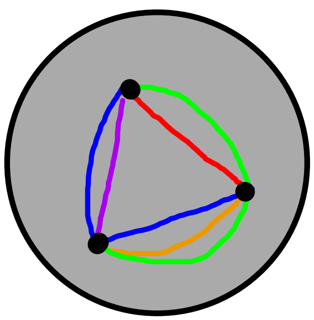
Step 3:

either



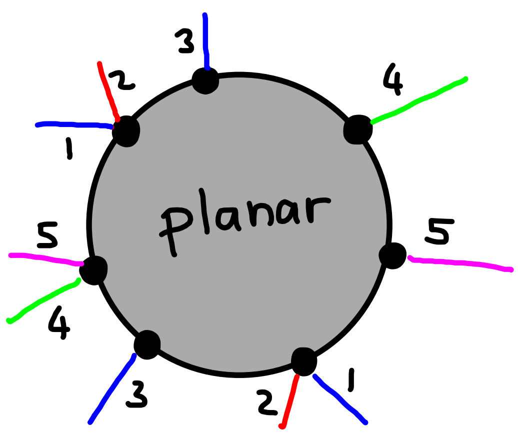
or at least 2 indep. edges in U

Proof idea: $\Omega = (G, \mathcal{B})$ biased graph with no two disjoint unbalanced cycles.

Steps 0, 1, 2, 3: G is 4-connected and Ω has a 2-connected spanning balanced subgraph Ω'. Pick Ω' **maximal**.

Set $U = E(\Omega) - E(\Omega')$. U has at least two indep. edges.

Step 4:



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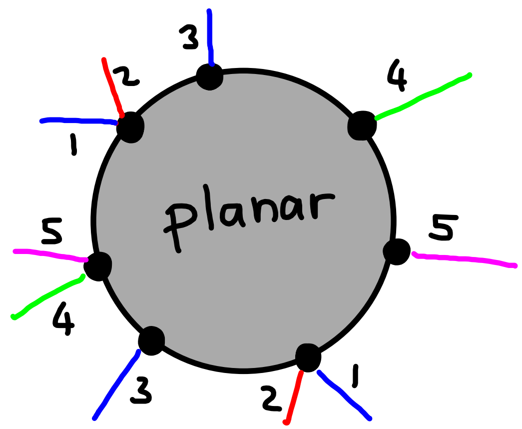
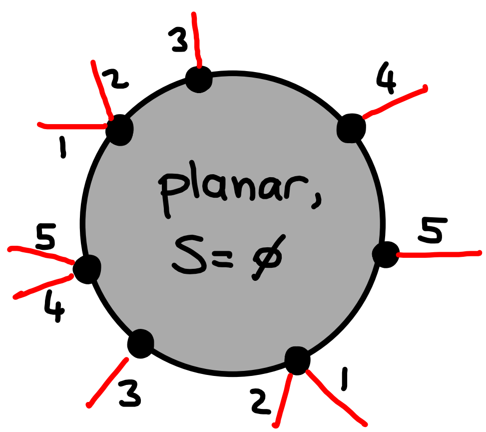
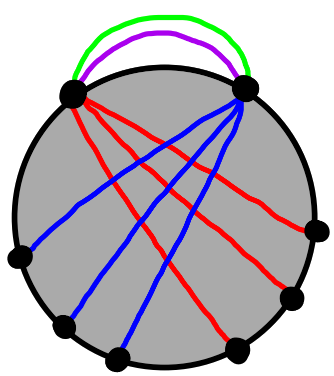




(projective planar signed graphic)

or



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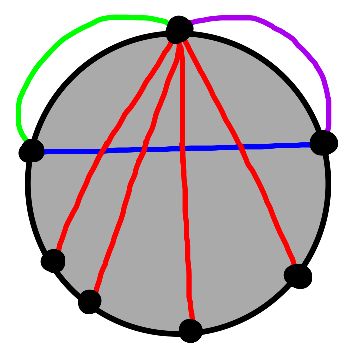

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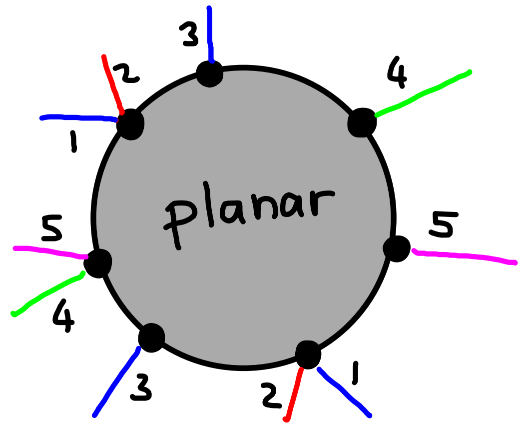

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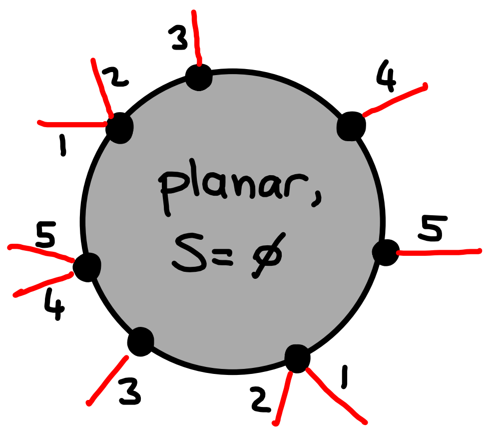

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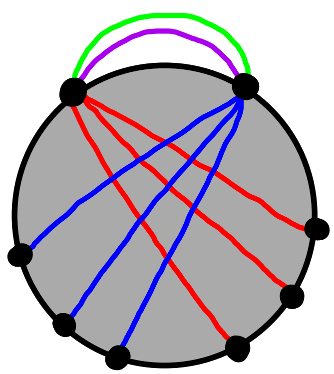

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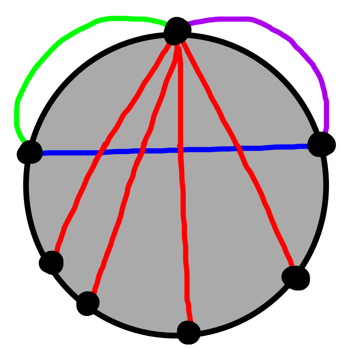

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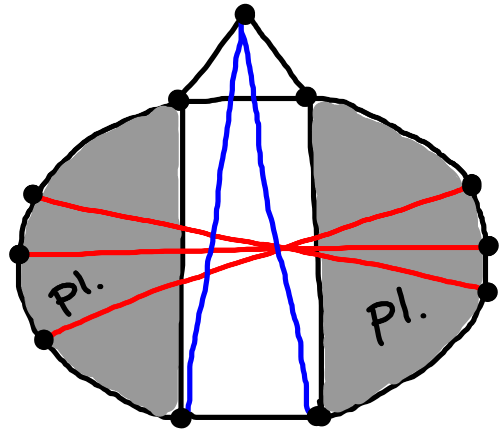

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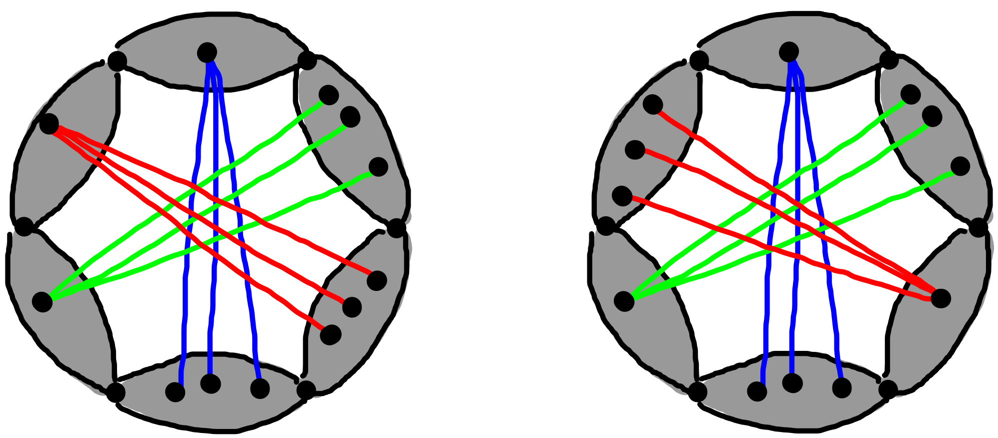

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THANK YOU!