# Binary and Regular Matroids without a Certain Minor.

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An $H$-free graph is a graph without an $H$-minor

## Graph Minors

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- The Petersen-graph has fifteen edges
- It is important problem to characterize $H$-free graphs with few edges

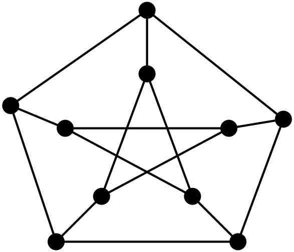

Figure: Petersen Graph

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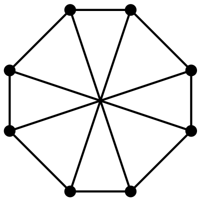

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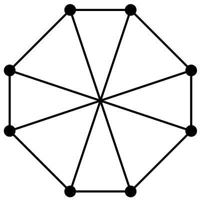

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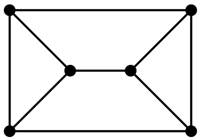

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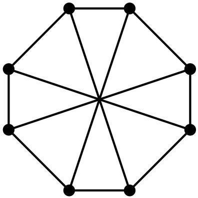

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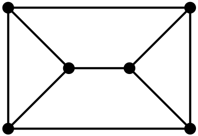

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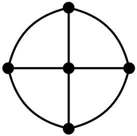

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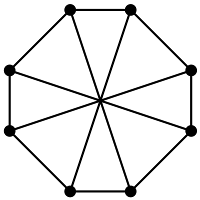

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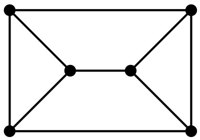

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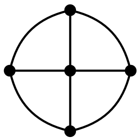

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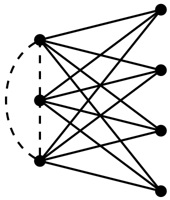
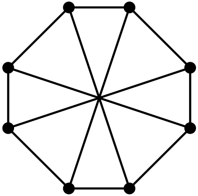


$K_{3,n}$-like



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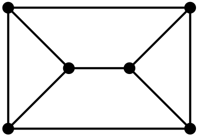

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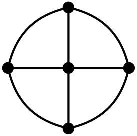

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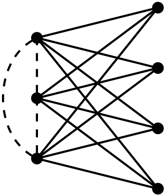

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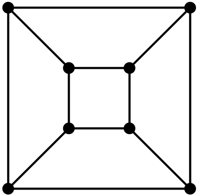
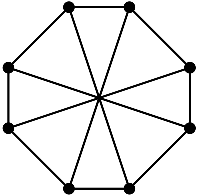


Cube



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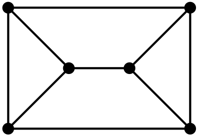

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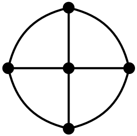

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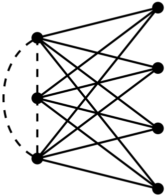

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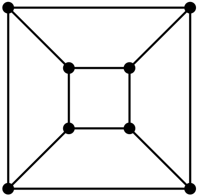

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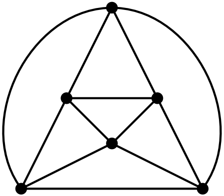

Octahedron

# Harville, Reid, and Wu, Annals of Combinatorics, to appear

## Lemma

*If $M_1$ and $M_2$ are binary matroids with at least six elements, $M_1$ is 3-connected, $M_2$ is connected, $E(M_1) \cap E(M_2) = T$, the set $T$ is a triangle of both, and neither $M_1$ nor $M_2$ has a cocircuit contained in $T$, then $M_1 \oplus_3 M_2$ is 2-connected.*

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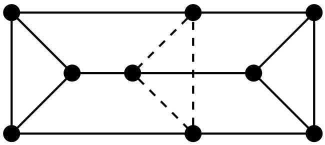
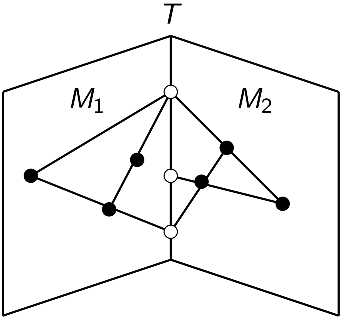


A 3-sum of two prisms

## Theorem (HRW)

*Let $N$ be a simple binary vertically 4-connected matroid with rank at least four. If $M$ is the 3-sum of binary $N$-free matroids $M_1$ and $M_2$, then $M$ is $N$-free.*

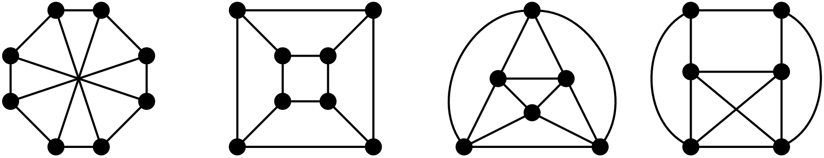



$P_T(M_1, M_2)$

### Theorem (Decomposition Theorem)

*Let $N$ be a simple, vertically 4-connected matroid with rank exceeding three. Then $M$ is a regular $N$-free matroid if and only if $M$ can be constructed by direct sums, 2-sums, or 3-sums starting with $N$-free matroids, each of which is isomorphic to a minor of $M$ and each of which is graphic, cographic, or is isomorphic to $R_{10}$ (if $R_{10}$ is $N$-free).*

(HRW) The 3-connected regular $N$-free matroids are characterized for $N \cong$



$V_8$ Cube Octahedron $K_5^{\perp}$

**Theorem**

*A 3-connected regular matroid $M$ is $\{M(W_5+e),\ M^*(W_5+e)\}$-free if and only if $M$ is one of the following matroids:*

(i) *the cycle matroid of a graph $G$ that is a member of $\mathcal{K}$, $\mathcal{W}$, or a 3-connected minor of $V_8$, cube, octahedron, pyramid, or $K_5^{\perp}$,*

(ii) *the dual matroid of a graph in (i),*

(iii) $R_{10}$, *or*

(iv) $R_{12}$.

## Theorem (Chun, Mayhew, and Oxley 2011)

*Let $M$ be an internally 4-connected binary matroid with at least seven elements. Then $M$ has a proper internally 4-connected minor $N$ with $|E(M)| - |E(N)| \leq 3$ unless $M$ or its dual is the cycle matroid of a planar or Möbius quartic ladder, or a terrahawk....*

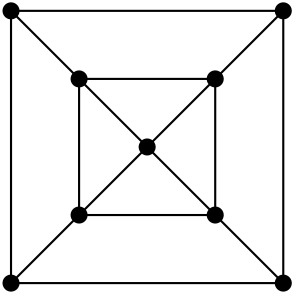


Figure: The Terrahawk

Theorem ( Mayhew and Royle, SIAM J. of Discrete Math., 2012)

*Let $M$ be a binary matroid with no prism-minor.*

1. *If $M$ is internally 4-connected, then $M$ has rank at most five, and is isomorphic to a minor of the apex $AG(3,2)$ matroid.*
2. *If $M$ is 3-connected, then ...*

This characterization uses the computer program MACEK of Petr Hlinӑný

**Theorem (Kingan and Lemos, 2012)**

*Suppose $M$ is a 3-connected binary non-regular matroid with no prism-minor. Then one of the following holds:*

(i) *$M$ is isomorphic to $Z_r$, $Z_r^*$, $Z_r \backslash y_r$, $Z_r \backslash t$ for some $r \geq 4$,*

(ii) *$P_9$ is a 3-decomposer for $M$,*

(iii) *$M$ is isomorphic to $(P_\triangle(F_7, F_7)\backslash z)^*$, or*

(iv) *$M$ has rank at most five.*

**Theorem (HRW)**

*An internally 4-connected binary matroid $M$ has no (prism+e)-minor if and only if $M$ is one of ninety matroids in the set $EX_{i4c}(prism + e)$. Each such matroid is an internally 4-connected minor of at least one of the matroids $P_{17}$, $P^*_{17}$, $Q_{15}$, $Q^*_{15}$, or $Q_{12}$; has at most 17 elements; and has rank or corank at most 5 with the exception of $Q_{12}$, where $Q_{12}$ is a 12-element matroid in $EX_{i4c}(prism + e)$ having rank and corank 6.*

# Some statistics on the class of internally 4-connected binary $T$-free matroids

- Each such matroid has at most 17 elements.
- All but one such matroid has rank or corank at most five.
- There is one such matroid with twelve elements and rank and corank 6.
- In this set of 90 matroids, 42 of them have no prism minors, and 48 of them have prism minors, but have no $T$- minor.
- There are 4 maximal matroids in the class. One of these is from Mayhew and Royle's result, the other three have twelve, fifteen, and seventeen elements.

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
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| $M(K_5)-1$ | ● | ● | | | | | | | ● | | | | | | ● | | | ● | ● | |
| $M7-1$ | | | | ● | ● | | | ● | | | | | | ● | | ● | ● | | ● | |
| $M7-2$ | | | | ● | ● | | | | | | | | | ● | ● | ● | ● | | ● | |
| $M7-3$ | | | | | ● | | | ● | | ● | | | | ● | | ● | ● | | | ● |
| $M8-1$ | ● | | ● | | ● | | ● | | | ● | | | | ● | | | | | | ● |
| $M9-1$ | ● | | | | ● | | ● | | | ● | ● | | | | ● | | | ● | ● | |
| $M10-1$ | | | | ● | ● | | | ● | ● | | ● | ● | | ● | | | | | ● | |
| $M10-2$ | | | ● | ● | ● | | | | | | | ● | | ● | ● | ● | | | ● | |
| $M10-3$ | | | | | | | | ● | ● | ● | ● | ● | ● | ● | | | | | ● | |
| $M11-1$ | ● | | | ● | ● | | | ● | ● | | ● | ● | | ● | | | | | ● | |
| $M11-2$ | ● | | ● | | ● | ● | | ● | | ● | ● | | | ● | | | | | | ● |
| $M12-1$ | ● | | ● | ● | ● | ● | | ● | | | | | | ● | | ● | ● | | ● | |
| $PG(3,2)-1$ | ● | ● | | ● | ● | ● | | ● | ● | | ● | | | | | | ● | ● | ● | |

Figure: Rank 5 sporadic members of $EX_{3c-i4c}(prism+e)$