Orienting GF(4)-representable matroids

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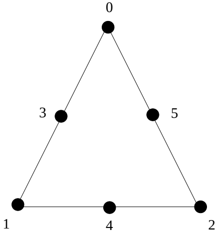
Oriented matroids from matrices over ordered fields

$$A = \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{bmatrix}_{\mathbb{R}}$$



Some signings:

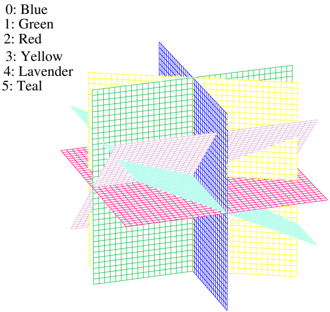
- $x = (0, -1, 1, 0, 1, 0)$ is an element of $NS(A)$ with minimal support. This gives rise to the signed circuit $(0, -, +, 0, +, 0)$
- $(2, -1, 0)A = (2, -1, 0, 0, -1, -2)$, so $\{2, 3\}$ is a flat of M. This gives rise to the signed covector $+-00--$.
  Geometry: A_2 and A_3 are the only columns in the hyperplane with normal $y = (2, -1, 0)$

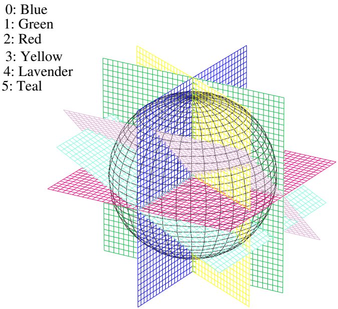
Covectors, Flats, and Cocircuits

- A covector indicates if a column vector from the matrix is on the positive side, on the negative side, or inside the hyperplane with normal y.
- Many normal vectors can produce the same covectors. For example, the normal vectors $y_1 = (0, 5, 1)$ and $y_2 = (0, 3, 1)$ both correspond to the covector $0 + + + ++$. So we can use covectors to partition $\mathbb{R}^3$.
- The support of a covector is the complement of a flat of the matroid.
- The support of a covector is the union of cocircuits.

Another view of covectors

Instead of thinking about columns vectors in planes, you can think of the column vectors as normals of planes through the the origin. Each plane has a positive side (in the direction of the normal) and a negative side. The signed regions represent the covectors.





Stand on your head

We will take one of the planes to be the equator and look at the top half of the sphere.

Lines are elements.

Points at which 3 or more lines meet are rank-2 flats.

In other words:

- Their Points $\leftrightarrow$ Our Lines
- Their Lines $\leftrightarrow$ Our Points

An Orientation of Q_6

$$A = \begin{array}{c} \\ \end{array}\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$

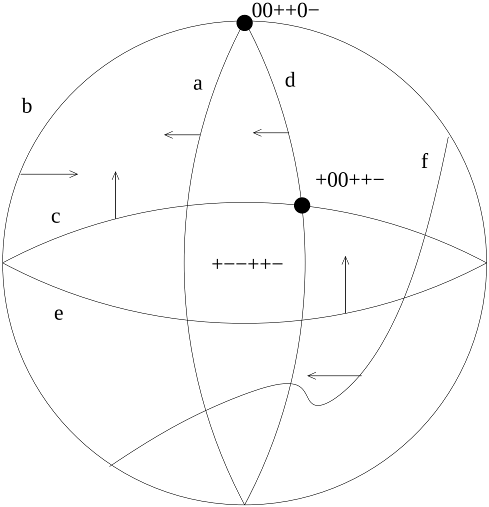
$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{\mathbb{R}}$$

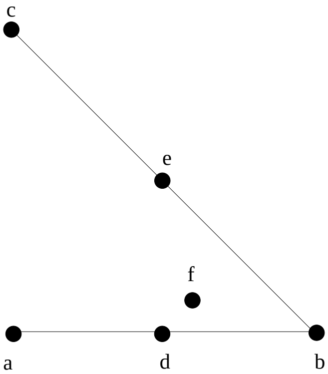
The affine points: (linear form $l(x) = x_1 + x_2 + x_3$)

$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} \end{bmatrix}_{\mathbb{R}}$$

An Orientation of Q_6





Axiom Systems

Definition 1 (Covector axioms for oriented matroids)

$\mathcal{M} = (E, \mathcal{F})$ is an oriented matroid on E if $\mathcal{F} \subseteq \{+, -, 0\}^E$ satisfies:

(F1) $0 \in \mathcal{F}$

(F2) If $X \in \mathcal{F}$ then $-X \in \mathcal{F}$

(F3) If $X, Y \in \mathcal{F}$ then $X \circ Y \in \mathcal{F}$

(F4) If $X, Y \in \mathcal{F}$ and $e \in D(X, Y)$, then $\exists Z \in \mathcal{F}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \backslash D(X, Y)$.

Axiom Systems

Proposition 2 (Circuit axioms for oriented matroids)

$\mathcal{M} = (E, \mathcal{C})$ *is an oriented matroid on* E *if* $\mathcal{C} \subseteq \{+, -, 0\}^E$ *satisfies:*

(C1) $0 \notin \mathcal{C}$

(C2) *If* $X \in \mathcal{C}$ *then* $-X \in \mathcal{C}$

(C3) *If* $X, Y \in \mathcal{C}$ *and* $\underline{X} \subseteq \underline{Y}$*, then* $X = Y$ *or* $X = -Y$

(C4) *If* $X, Y \in \mathcal{C}$, $X \neq -Y$*, and* $e \in D(X, Y)$*, then* $\exists Z \in \mathcal{C}$ *such that* $Z_e = 0$ *and* $Z^+ \subseteq (X^+ \cup Y^+) \backslash \{e\}$ *and* $Z^- \subseteq (X^- \cup Y^-) \backslash \{e\}$.

Some Examples

- Fano matroid is not orientable
- Maclane matroid $(\mathrm{AG}(2,3)\backslash \mathrm{p})$ is not orientable
- Vámos matroid is not representable but it is orientable

History

An orientation is regular, dyadic, or golden mean if the orientation can be derived from a matrix that is totally unimodular, dyadic, or totally golden mean, respectively.

Proposition 3 (Bland and Las Vergnas, 1978)

- *A binary matroid is orientable iff it is regular.*
- *Any orientation of a binary matroid is regular.*
- *If a binary matroid is orientable, it has only one orientation, up to reorientation.*

History

Proposition 4 (Lee and Scobee, 1999)

- *A ternary matroid is orientable iff it is dyadic.*
- *Any orientation of a ternary matroid is dyadic.*
- *If a ternary matroid is orientable, it has at most 3 orientations, up to reorientation.*

The logical question

Question 5

Can we say something nice about the representations of matroids that are representable over $\mathrm{GF}(4)$ *and orientable?*

The disappointing answer

The rank-k free spike without a tip.

Elements: $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$

Circuits:

- $\{a_i, b_i, a_j, b_j\}$ where $i \neq j$
- $\{a_i, b_i, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ where $x_j \in \{a_j, b_j\}$

Orientations of the free spikes

The free spikes without a tip are:

- $\mathrm{GF}(4)$-representable
- orientable

Unfortunately, some of their orientations are not realizable.

Boolean functions

Rank-2 boolean functions

00	01	10	11
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1

00	01	10	11
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

Monotone boolean functions

A boolean function is monotone if $f(Y) = 0 \implies f(X) = 0$ whenever $X \preceq Y$ (in the lexicographic sense, not the binary number sense.) D_k is the number of rank-k monotone boolean functions.

- Rank-0: 0, 1 : $D_0 = 2$
- Rank-1: 00, 01, 11 : $D_1 = 3$
- Rank-2: 0000, 0001, 0011, 0101, 0111, 1111 : $D_2 = 6$
- Rank-3: $D_3 = 20$
- Rank-4: $D_4 = 168$
- Rank-5: $D_5 = 7,581$
- Rank-6: $D_6 = 7,828,345$
- Rank-7: $D_7 = 2,414,682,040,998$
- Rank-8: $D_8 = 56,130,437,228,687,557,907,788$

Monotone boolean functions

Theorem 6

For $k \geq 4$, there are exactly $2^{k-1} D_k$ inequivalent orientations of the rank-k free-spike without a tip.

Threshold functions

Definition 7

Let $f(x_1x_2\ldots x_k)$ be a rank-k boolean function. Then f is a threshold function on at most k variables if there exist real weights $w_1,\ldots w_k$ and a real threshold t such that

$$\sum_{i=1}^{k} w_i x_i \geq t \iff f(x_1x_2\ldots x_k) = 1$$
$$\sum_{i=1}^{k} w_i x_i < t \iff f(x_1x_2\ldots x_k) = 0$$

for all $x = x_1x_2\ldots x_k \in \mathscr{B}^k$.

Threshold functions

Theorem 8

The number of representable orientations of the rank-k free spike without a tip is bounded above by $2^{k-1}|N_k|$ where N_k is the set of N-equivalence classes of threshold functions on at most k variables.

$|N_k| < D_k \implies$ there are orientations of a $\mathrm{GF}(p^k)$-representable matroids $(k \geq 2)$ that are not representable.

Threshold functions

OEIS A002078 (N-equivalence classes of threshold functions):

2, 3, 6, 20, 150, 3287, 244158, 66291591, 68863243522

OEIS A000372 (Monotone boolean functions, Dedekind numbers)

2, 3, 6, 20, 168, 7581, 7828354, 2414682040998,
56130437228687557907788

What can we hope for?

That somehow orientable and quaternary without certain minors implies that the matroid is representable over a nice partial field matrix. In particular, we are hoping to say:

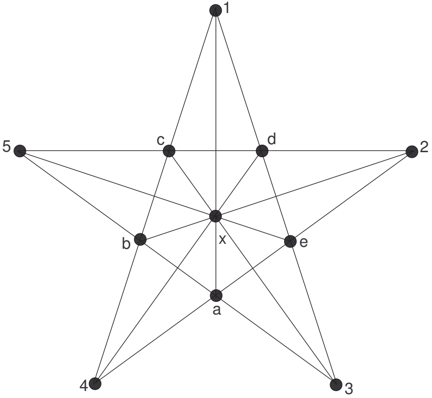
Question 9

Is it possible to find a set of matroids X such that if M is quaternary and orientable and does not contain a minor in X then M is golden mean.

Question 10

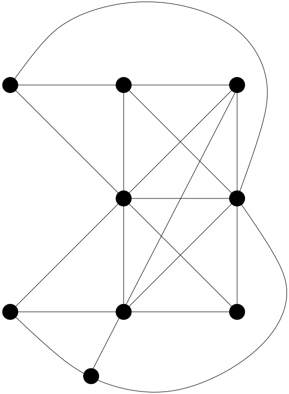
Is it possible to find a set of matroids Y such that if M is quaternary, orientable, and contains a minor in Y, then every orientation of M is golden mean?

Betsy Ross Matroid



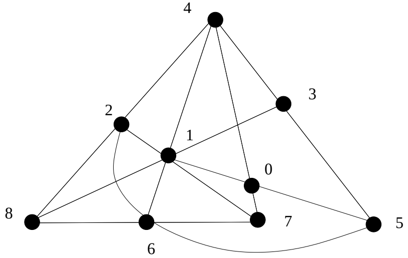
- Introduced by Brylawski and Kelly
- Only maximum-sized rank-3, golden-mean matroid (Archer, 2005)
- Orientable and all orientations are golden mean

$Q_{5,9}$



- $Q_{5,9}$ is representable over $\mathrm{GF}(4)$ but not $GF(5)$.
- $Q_{5,9}$ is not orientable because the MacLane matroid $(AG(2,3)\backslash p)$ is a minor.

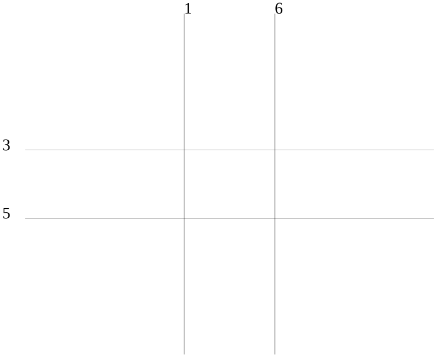
$Q_{11,9}$



- $Q_{11,9}$ is representable over $\mathrm{GF}(4)$ but not $GF(5)$.
- $Q_{11,9}$ does not contain MacLane matroid $(AG(2,3)\backslash p)$ minor or a Fano minor.

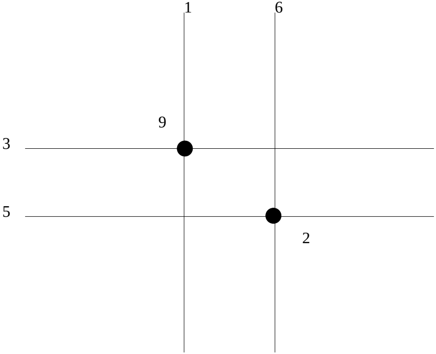
$Q_{11,9}$

The line at infinity represents 4.



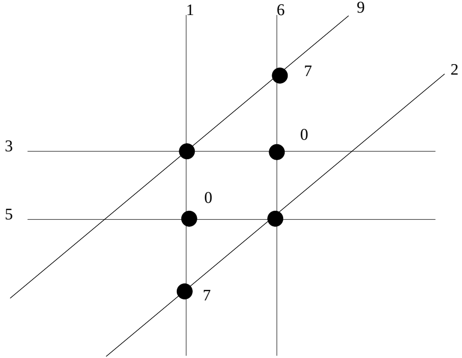
$Q_{11,9}$

The line at infinity represents 4.



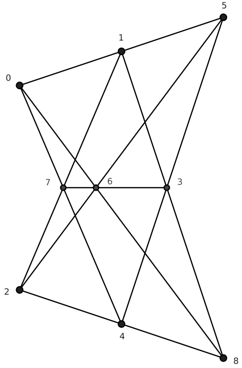
$Q_{11,9}$

The line at infinity represents 4.



- $Q_{11,9}$ is an excluded minor for golden mean matroids.
- $Q_{11,9}$ is an excluded minor for orientable matroids.

Pappus matroid

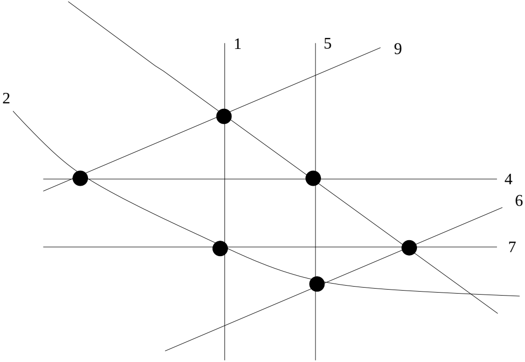


The Pappus matroid:

- is representable over $\mathrm{GF}(4)$ but not $GF(5)$
- is an excluded minor for golden mean matroids

Pappus matroid

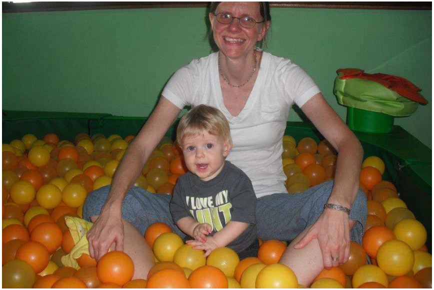
The line at infinity represents 0.



The Pappus matroid is orientable.

The End

The negative side



The positive side