Enumeration of 2-Polymatroids on up to Seven Elements

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Definition of a $k$-polymatroid

**Definition**
Let $S$ be a finite set. Suppose $\rho : 2^S \to \mathbb{N}$ satisfies the following four conditions:

(i) if $X, Y \subseteq S$, then $\rho(X \cap Y) + \rho(X \cup Y) \leq \rho(X) + \rho(Y)$ (submodular);
(ii) if $X \subseteq Y \subseteq S$, then $\rho(X) \leq \rho(Y)$ (monotone);
(iii) $\rho(\emptyset) = 0$ (normalized); and
(iv) $\rho(\{x\}) \leq k$ for all $x \in S$.

Then $(\rho, S)$ is a $k$-polymatroid with rank function $\rho$ and ground set $S$.

A matroid is a 1-polymatroid.
Theorem (Helgason 1974, McDiarmid 1975, Lovász 1977)

Let \((\rho, S)\) be a \(k\)-polymatroid. Then there exists a matroid \((r, E)\) and a map \(\sigma: S \to 2^E\) such that \(\rho(X) = r(\bigcup_{x \in X} \sigma(x))\) for all \(X \subseteq S\).
Examples of 2-polymatroids

Let \((r_1, E)\) and \((r_2, E)\) be matroids. Then \((r_1 + r_2, E)\) is a 2-polymatroid.
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Let \(G = (V, E)\) be a graph. For \(X \subseteq E\), define \(\rho(X)\) as the number of vertices incident to some edge in \(X\). Then \((\rho, E)\) is a 2-polymatroid. For example:

\[
\begin{align*}
\rho(a) &= 2 \\
\rho(abc) &= 3 \\
\rho(ad) &= 4
\end{align*}
\]
A flat of a $k$-polymatroid $(\rho, S)$ is a set $F \subseteq S$ with the property that $\rho(F \cup e) > \rho(F)$ for each $e \in S - F$.

Knowledge of the flats and their ranks of a $k$-polymatroid suffices to determine the entire $k$-polymatroid.

As for matroids, the intersection of flats is itself a flat.
Definition

Let \((\rho, S)\) be a polymatroid, and let \(e\) be an element not in \(S\). If \((\bar{\rho}, S \cup e)\) is a polymatroid with \(\bar{\rho}(X) = \rho(X)\) for all \(X \subseteq S\), then \(\bar{\rho}\) is a single-element extension of \(\rho\).
Definition
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Let \(c = \bar{\rho}(e)\). Partition the flats of \((\rho, S)\) into classes

\[
\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_c
\]

by the rule \(F \in \mathcal{M}_i\) if and only if \(\bar{\rho}(F \cup e) = \rho(F) + i\).

(Note that some \(\mathcal{M}_i\) may be empty.)
Modular Cuts

Definition
A modular cut of a polymatroid \((\rho, S)\) is a subset \(M\) of \(F(\rho, S)\) which is closed under:

(i) supersets and
(ii) intersections of modular pairs.

Theorem (Crapo 1964)
The single-element extensions of a matroid are in one-to-one correspondence with its modular cuts.

The key point is that \(M_0\) is a modular cut.
Example 1
Example 2

\[ a \quad \| \quad b \]

\[ a \quad \| \quad b \quad c \]

\[ a \quad b \quad \emptyset \quad M_0 \quad M_2 \]
Example 3
Example 4

\[ a \quad \rightarrow_{b} \quad b \]

\[ a \quad \rightarrow_{c} \quad b \quad \text{rank 4} \]

\[ a \quad \underset{\varnothing}{\rightarrow} \quad b \]

\[ a \quad \underset{\varnothing}{\rightarrow} \quad b \quad \text{rank 4} \]

\[ M_1 \quad \text{and} \quad M_2 \]
Extensible Partitions

Which partitions $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_c$ give rise to single-element extensions of the $k$-polymatroid $(\rho, S)$?
Extensible Partitions

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Define $\bar{\rho}: 2^{S \cup e} \to \mathbb{N}$ as follows.

- For $X \subseteq S$, set $\bar{\rho}(X) = \rho(X)$.
- If $\text{cl}(X) \in \mathcal{M}_i$, then set $\bar{\rho}(X \cup e) = \rho(X) + i$. 


Extensible Partitions

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- If $\text{cl}(X) \in M_i$, then set $\bar{\rho}(X \cup e) = \rho(X) + i$.

Recall the definition of modular defect:

$$\delta(X, Y) = \rho(X) + \rho(Y) - \rho(X \cup Y) - \rho(X \cap Y).$$

Define also

$$\mu(X) = i \quad \text{if and only if} \quad \text{cl}(X) \in M_i.$$
A Characterization of Extensible Partitions

**Theorem (Savitsky 2014)**

As defined above, $(\tilde{\rho}, S \cup e)$ is a polymatroid, and hence a single-element extension of $(\rho, S)$, if and only if the following three conditions hold for all flats $F, G$ of $(\rho, S)$:

(I) $\mu(F \cap G) + \mu(F \cup G) - \delta(F, G) \leq \mu(F) + \mu(G)$;

(II) if $F \subseteq G$, then $\rho(F) + \mu(F) \leq \rho(G) + \mu(G)$; and

(III) if $F \subseteq G$, then $\mu(G) \leq \mu(F)$. 
Theorem (Savitsky 2014)

As defined above, \((\bar{\rho}, S \cup e)\) is a polymatroid, and hence a single-element extension of \((\rho, S)\), if and only if the following three conditions hold for all flats \(F, G\) of \((\rho, S)\):

(I) \(\mu(F \cap G) + \mu(F \cup G) - \delta(F, G) \leq \mu(F) + \mu(G)\);
(II) if \(F \subseteq G\), then \(\rho(F) + \mu(F) \leq \rho(G) + \mu(G)\); and
(III) if \(F \subseteq G\), then \(\mu(G) \leq \mu(F)\).

The flats of a single-element extension of a \(k\)-polymatroid may be described in terms of the flats of the original \(k\)-polymatroid.
Can we construct a catalog of all non-isomorphic $k$-polymatroids on the ground set $\{1, \ldots, n\}$? Presumably $n$ and $k$ must be small.

Much work has been done for the matroid case ($k = 1$) with the aid of computers.
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A Brief History of Matroid Enumeration

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In 2007, Mayhew and Royle created a database of all matroids on at most 9 elements.
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In 2007, Mayhew and Royle created a database of all matroids on at most 9 elements.

In 2012, Matsumoto, Moriyama, Imai, and Bremner constructed all 10-element matroids with rank not equal to 5.
The Number of Small Matroids by Rank

This table lists the number of non-isomorphic matroids on the ground set \{1, \ldots, n\} by rank.

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Having developed an analogue of the theory of single-element extensions of matroids, I managed to adapt Mayhew and Royle’s enumeration approach to 2-polymatroids.

Using a desktop computer, I created a catalog of all non-isomorphic 2-polymatroids on at most 7 elements. This computation took about 4 days.

The results are summarized on the next slide, which lists the number of non-isomorphic 2-polymatroids on the ground set \( \{1, \ldots, n\} \) by rank.
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The Number of (Unlabeled) Small 2-polymatroids by Rank

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</table>
Non-unimodality

The rank of a $k$-polymatroid $(\rho, S)$ is $\rho(S)$.

Surprisingly, 2-polymatroids on 7 elements are not unimodal in rank. Since there are fewer of rank 7 than of rank 6, there is a “dip” in the middle.
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The following notion of $k$-duality for $k$-polymatroids explains the symmetry in the columns:

$$\rho^*(X) = k|X| + \rho(S - X) - \rho(S).$$
Linear Inequalities

Recall the definition of a $k$-polymatroid.

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(iv) $\rho(\{x\}) \leq k$ for all $x \in S$.

Now think of the $2^{|S|}$ values $\rho(X)$ as variables. Notice that all the inequalities they must satisfy are **linear**.
Therefore, the feasible region of an integer program gives all \( k \)-polymatroids on a fixed ground set. Many of these will of course be isomorphic.

The objective function of the integer program is immaterial. We want all solutions, not just “optimal” ones.
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Using the following condition equivalent to submodularity works better in practice. For $A \subseteq S$ and $f, g \in S - A$,

$$\rho(A) + \rho(A \cup f \cup g) \leq \rho(A \cup f) + \rho(A \cup g).$$
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Parallelism is easily achieved by splitting the computation into many small integer programs, each extending a single polymatroid.
Labeled versus Unlabeled Polymatroids

The catalog, however, contains unlabeled 2-polymatroids; i.e., it has one representative of each isomorphism class of 2-polymatroids.

Let $(\rho, \{1, \ldots, n\})$ be a polymatroid. By the Orbit-Stabilizer Relation, there are \(\frac{n!}{|Aut(\rho)|}\) isomorphic copies of \(\rho\) on the same ground set.
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Let \((\rho, \{1, \ldots, n\})\) be a polymatroid. By the Orbit-Stabilizer Relation, there are \(\frac{n!}{|\text{Aut}(\rho)|}\) isomorphic copies of \(\rho\) on the same ground set.

Fortunately, computing the automorphism group of a polymatroid can be reduced to computing the automorphism group of a (colored) graph. This can be found quickly with Brendan McKay’s \text{nauty} program.

The table in the next slide lists the number of labeled 2-polymatroids on the ground set \(\{1, \ldots, n\}\).
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</table>
The open-source optimization suite SCIP (http://scip.zib.de/) is able to efficiently count the number of feasible solutions to integer programs. It took about 13 weeks, using SCIP, to verify the numbers in the previous table.

Important Technical Note: It was necessary to turn off all presolving options in SCIP in order to obtain accurate counts.

These two different methods of counting 2-polymatroids agree exactly.
Conjectures

Conjecture

Almost all $k$-polymatroids contain no element of rank less than $k$.

This would generalize the theorem that almost all matroids are loopless by Mayhew, Newman, Welsh, and Whittle (2011).

Conjecture

Almost all $k$-polymatroids are asymmetric.