Enumeration of 2-Polymatroids on up to Seven Elements

Thomas J. Savitsky

The George Washington University

July 21, 2014

Definition of a k-polymatroid

Definition

Let S be a finite set. Suppose $\rho\colon 2^S \to \mathbb{N}$ satisfies the following four conditions:

(i) if $X, Y \subseteq S$, then $\rho(X \cap Y) + \rho(X \cup Y) \leq \rho(X) + \rho(Y)$ (submodular);

(ii) if $X \subseteq Y \subseteq S$, then $\rho(X) \leq \rho(Y)$ (monotone);

(iii) $\rho(\varnothing) = 0$ (normalized); and

(iv) $\rho(\{x\}) \leq k$ for all $x \in S$.

Then (ρ, S) is a k-polymatroid with rank function ρ and ground set S.

A matroid is a 1-polymatroid.

Representing k-polymatroids with Matroids

Theorem (Helgason 1974, McDiarmid 1975, Lovász 1977)

Let (ρ, S) be a k-polymatroid. Then there exists a matroid (r, E) and a map $\sigma\colon S \to 2^E$ such that $\rho(X) = r(\bigcup_{x \in X} \sigma(x))$ for all $X \subseteq S$.

Examples of 2-polymatroids

Let (r_1, E) and (r_2, E) be matroids. Then $(r_1 + r_2, E)$ is a 2-polymatroid.

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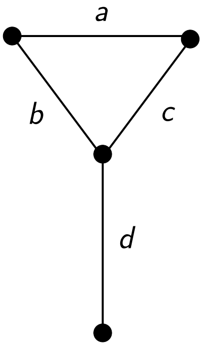
Let $G = (V, E)$ be a graph. For $X \subseteq E$, define $\rho(X)$ as the number of vertices incident to some edge in X. Then (ρ, E) is a 2-polymatroid. For example:



$\rho(a) = 2$ $\rho(abc) = 3$ $\rho(ad) = 4$

Flats

A flat of a k-polymatroid (ρ, S) is a set $F \subseteq S$ with the property that $\rho(F \cup e) > \rho(F)$ for each $e \in S - F$.

Knowledge of the flats and their ranks of a k-polymatroid suffices to determine the entire k-polymatroid.

As for matroids, the intersection of flats is itself a flat.

Single-Element Extensions of k-Polymatroids

Definition

Let (ρ, S) be a polymatroid, and let e be an element not in S. If $(\bar{\rho}, S \cup e)$ is a polymatroid with $\bar{\rho}(X) = \rho(X)$ for all $X \subseteq S$, then $\bar{\rho}$ is a single-element extension of ρ.

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Let $c = \bar{\rho}(e)$. Partition the flats of (ρ, S) into classes

$$\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_c$$

by the rule $F \in \mathcal{M}_i$ if and only if $\bar{\rho}(F \cup e) = \rho(F) + i$.

(Note that some $\mathcal{M}_i$ may be empty.)

Modular Cuts

Definition

A modular cut of a polymatroid (ρ, S) is a subset $\mathcal{M}$ of $\mathcal{F}(\rho, S)$ which is closed under:

(i) supersets and

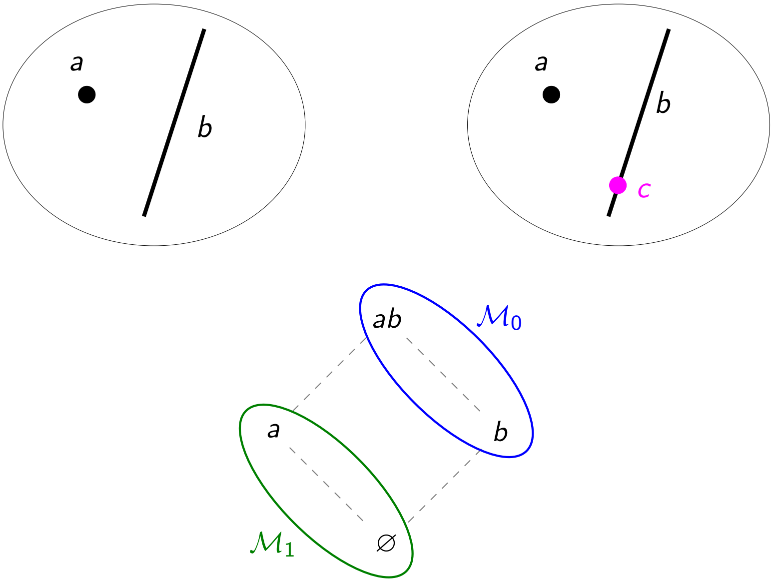
(ii) intersections of modular pairs.

Theorem (Crapo 1964)

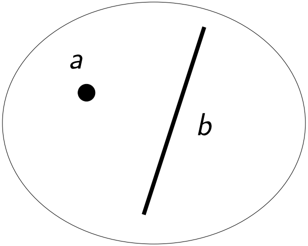
The single-element extensions of a matroid are in one-to-one correspondence with its modular cuts.

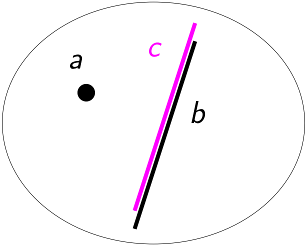
The key point is that $\mathcal{M}_0$ is a modular cut.

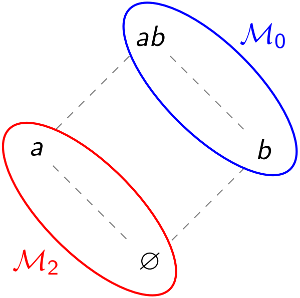
Example 1



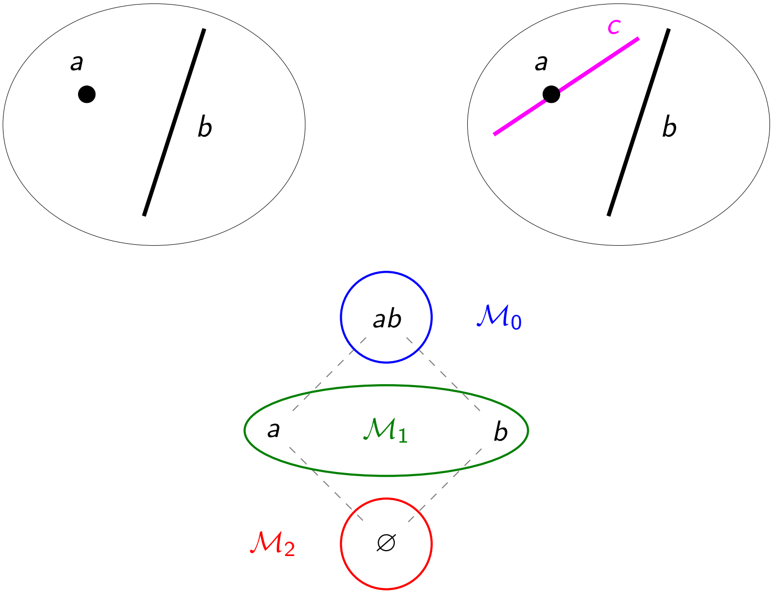
Example 2



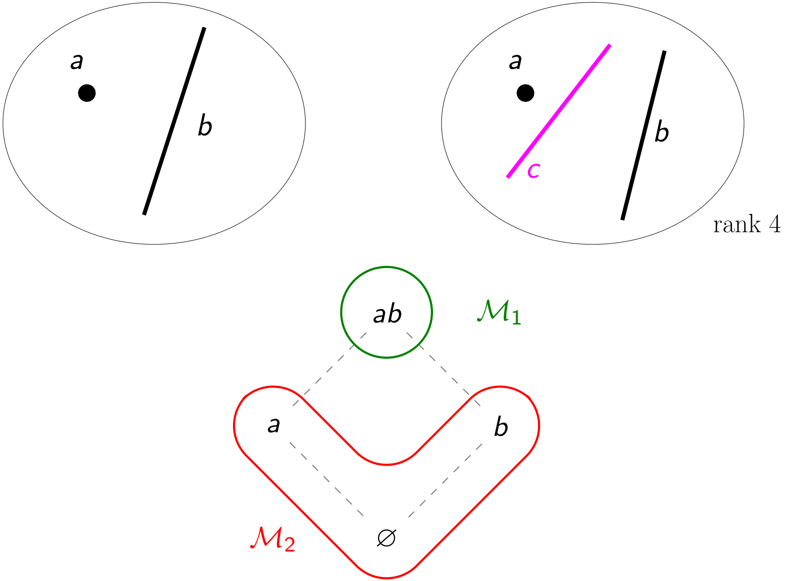




Example 3



Example 4



Extensible Partitions

Which partitions $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_c$ give rise to single-element extensions of the k-polymatroid (ρ, S)?

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Define $\bar{\rho}$: $2^{S \cup e} \to \mathbb{N}$ as follows.

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Recall the definition of modular defect:

$$\delta(X, Y) = \rho(X) + \rho(Y) - \rho(X \cup Y) - \rho(X \cap Y).$$

Define also

$$\mu(X) = i \quad \text{if and only if} \quad \mathrm{cl}(X) \in \mathcal{M}_i.$$

A Characterization of Extensible Partitions

Theorem (Savitsky 2014)

As defined above, $(\bar{\rho}, S \cup e)$ is a polymatroid, and hence a single-element extension of (ρ, S), if and only if the following three conditions hold for all flats F, G of (ρ, S):

(I) $\mu(F \cap G) + \mu(F \cup G) - \delta(F, G) \leq \mu(F) + \mu(G)$;

(II) *if* $F \subseteq G$, *then* $\rho(F) + \mu(F) \leq \rho(G) + \mu(G)$; *and*

(III) *if* $F \subseteq G$, *then* $\mu(G) \leq \mu(F)$.

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The flats of a single-element extension of a k-polymatroid may be described in terms of the flats of the original k-polymatroid.

A Brief History of Matroid Enumeration

Can we construct a catalog of all non-isomorphic k-polymatroids on the ground set $\{1, \ldots, n\}$? Presumably n and k must be small.

Much work has been done for the matroid case ($k = 1$) with the aid of computers.

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In 2012, Matsumoto, Moriyama, Imai, and Bremner constructed all 10-element matroids with rank not equal to 5.

The Number of Small Matroids by Rank

This table lists the number of non-isomorphic matroids on the ground set $\{1, \ldots, n\}$ by rank.

rank \ n	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10
2		1	3	7	13	23	37	58	87	128
3			1	4	13	38	108	325	1275	10037
4				1	5	23	108	940	190,214	4,886,380,924
5					1	6	37	325	190,214	$> 2.6 \times 10^{12}$
6						1	7	58	1275	4,886,380,924
7							1	8	87	10037
8								1	9	128
9									1	10
10										1
total	2	4	8	17	38	98	306	1724	383,172	$> 2.6 \times 10^{12}$

A Catalog of Small 2-polymatroids

Having developed an analogue of the theory of single-element extensions of matroids, I managed to adapt Mayhew and Royle's enumeration approach to 2-polymatroids.

Using a desktop computer, I created a catalog of all non-isomorphic 2-polymatroids on at most 7 elements. This computation took about 4 days.

The results are summarized on the next slide, which lists the number of non-isomorphic 2-polymatroids on the ground set $\{1, \ldots, n\}$ by rank.

The Number of (Unlabeled) Small 2-polymatroids by Rank

rank \ n	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1		1	2	3	4	5	6
2		1	4	10	21	39	68
3			2	12	49	172	573
4			1	10	78	584	5236
5				3	49	778	18,033
6				1	21	584	46,661
7					4	172	18,033
8					1	39	5236
9						5	573
10						1	68
11							6
12							1
13							
14							
total	1	3	10	40	228	2380	94,495

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rank \ n	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7
2		1	4	10	21	39	68	112
3			2	12	49	172	573	1890
4			1	10	78	584	5236	72,205
5				3	49	778	18,033	971,573
6				1	21	584	46,661	149,636,721
7					4	172	18,033	19,498,369
8					1	39	5236	149,636,721
9						5	573	971,573
10						1	68	72,205
11							6	1890
12							1	112
13								7
14								1
total	1	3	10	40	228	2380	94,495	320,863,387

Non-unimodality

The rank of a k-polymatroid (ρ, S) is $\rho(S)$.

Surprisingly, 2-polymatroids on 7 elements are not unimodal in rank. Since there are fewer of rank 7 than of rank 6, there is a "dip" in the middle.

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The following notion of k-duality for k-polymatroids explains the symmetry in the columns:

$$\rho^*(X) = k|X| + \rho(S - X) - \rho(S).$$

Linear Inequalities

Recall the definition of a k-polymatroid.

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Now think of the $2^{|S|}$ values $\rho(X)$ as variables. Notice that all the inequalities they must satisfy are linear.

An Integer Programming Approach

Therefore, the feasible region of an integer program gives all k-polymatroids on a fixed ground set. Many of these will of course be isomorphic.

The objective function of the integer program is immaterial. We want all solutions, not just “optimal” ones.

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Using the following condition equivalent to submodularity works better in practice. For $A \subseteq S$ and $f, g \in S - A$,

$$\rho(A) + \rho(A \cup f \cup g) \leq \rho(A \cup f) + \rho(A \cup g).$$

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Parallelism is easily achieved by splitting the computation into many small integer programs, each extending a single polymatroid.

Labeled versus Unlabeled Polymatroids

The catalog, however, contains unlabeled 2-polymatroids; i.e., it has one representative of each isomorphism class of 2-polymatroids.

Let $(\rho, \{1, \ldots, n\})$ be a polymatroid. By the Orbit-Stabilizer Relation, there are $\frac{n!}{|Aut(\rho)|}$ isomorphic copies of ρ on the same ground set.

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Fortunately, computing the automorphism group of a polymatroid can be reduced to computing the automorphism group of a (colored) graph. This can be found quickly with Brendan McKay's `nauty` program.

The table in the next slide lists the number of labeled 2-polymatroids on the ground set $\{1, \ldots, n\}$.

rank \ n	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1		1	3	7	15	31	63	127
2		1	6	29	135	642	3199	16879
3			3	41	477	5957	87477	1604768
4			1	29	784	27375	1554077	189213842
5				7	477	41695	7109189	3559635761
6				1	135	27375	21937982	733133160992
7					15	5957	7109189	86322358307
8					1	642	1554077	733133160992
9						31	87477	3559635761
10						1	3199	189213842
11							63	1604768
12							1	16879
13								127
14								1
total	1	3	14	115	2040	109707	39445994	1560089623047

SCIP

The open-source optimization suite SCIP (`http://scip.zib.de/`) is able to efficiently count the number of feasible solutions to integer programs. It took about 13 weeks, using SCIP, to verify the numbers in the previous table.

Important Technical Note: It was necessary to turn off all presolving options in SCIP in order to obtain accurate counts.

These two different methods of counting 2-polymatroids agree exactly.

Conjectures

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Almost all k-polymatroids contain no element of rank less than k.

This would generalize the theorem that almost all matroids are loopless by Mayhew, Newman, Welsh, and Whittle (2011).

Conjecture

Almost all k-polymatroids are asymmetric.