Mock-threshold graphs

Vaidy Sivaraman

Department of Mathematical Sciences, SUNY (Binghamton)

(joint work with Richard Behr and Thomas Zaslavsky)

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Outline

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All graphs in this talk are finite and simple.

Let k be a non-negative integer. Let $\mathcal{G}_k$ denote the class of graphs that can be constructed from K_1 by repeatedly adding a vertex with at most k neighbors or at most k non-neighbors.

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Problem: Determine *Forb*($\mathcal{G}_k$).

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- *Threshold* $\subset$ *Split* $\subset$ *Chordal* $\subset$ *Weakly Chordal* $\subset$ *Perfect*

Berge defined the class of perfect graphs and offered two conjectures, both destined to become classic theorems.

Theorem (Perfect Graph Theorem - Lovász, 1972)

A graph is perfect if and only if its complement is perfect.

A hole in a graph is an induced cycle of length at least four. An antihole in a graph is an induced cycle of length at least four in the complement of the graph.

Theorem (Strong Perfect Graph Theorem - Chudnovsky, Robertson, Seymour, Thomas, 2006)

A graph is perfect if and only if contains neither an odd hole nor an odd antihole.

Corollary

Weakly chordal graphs are perfect.

Forbidden induced subgraph characterization of threshold graphs, split graphs and perfect graphs

Threshold $\subset$ *Split* $\subset$ *Chordal* $\subset$ *Weakly Chordal* $\subset$ *Perfect*

Theorem (Chvátal-Hammer 1973)

A graph is threshold if and only if it contains no induced subgraph isomorphic to $2K_2$, P_4, *or* C_4.

Theorem (Földes-Hammer 1977)

A graph is split if and only if it contains no induced subgraph isomorphic to $2K_2$, C_4, *or* C_5.

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Definition

A graph G is said to be **threshold** if there is a function $w : V(G) \to \mathbb{R}$ and a real number t such that there is an edge between two distinct vertices u and v if and only if $w(u) + w(v) > t$.

Consider a threshold graph G. Let v_{max} be a vertex with maximum weight and let v_{min} be a vertex with minimum weight. If the sum of their weights is greater than the threshold then v_{max} is a dominating vertex; else v_{min} is an isolated vertex. In fact,

A graph G is threshold if and only if G has a vertex ordering $v_1, \cdots, v_n$ such that for every i ($1 \leq i \leq n$) the degree of v_i in $G : \{v_1, \cdots, v_i\}$ is 0 or $i - 1$.

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Let us call a graph G mock-threshold if $G \in \mathcal{G}_1$.

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Thus $\mathcal{G}_0$ is the class of threshold graphs.

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In other words, a graph G is said to be **mock-threshold** if there is a vertex ordering $v_1, \cdots, v_n$ such that for every i ($1 \leq i \leq n$) the degree of v_i in $G : \{v_1, \cdots, v_i\}$ is $0, 1, i - 2$, or $i - 1$. Such an ordering will be called MT-ordering.

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Relax the definition without getting out of the class of perfect graphs.

As an easy consequence of the definition, we have the following.

Proposition

Let G be a graph on n vertices with $1 < \delta(G) \leq \Delta(G) < n - 2$. *Then G is not mock-threshold.*

The complement of a mock-threshold graph is also mock-threshold. Although the class of mock-threshold graphs is not closed under taking subgraphs, it is closed under taking induced subgraphs.
We are looking for a forbidden induced subgraph characterization for the class of mock-threshold graphs.

Proposition

A forest is a mock-threshold graph.

Proposition

K_n *and* $K_{2,n}$ *are mock-threshold.*

Proposition

Let $n \geq 5$. Then both C_n and $\overline{C_n}$ are minimal non-mock-threshold graphs.

Corollary

A mock-threshold graph is weakly chordal, and hence, perfect.

For a positive integer k, the k-core of a graph G is the graph obtained from G by repeatedly deleting vertices of degree less than k. It is routine to show that this is well-defined.

Proposition

A graph is mock-threshold if and only if its 2-core is also mock-threshold.

Graphs in $Forb(\mathcal{G}_1)$ with 5 and 6 vertices

Proposition

Every graph on at most five vertices except C_5 is mock-threshold.

Proposition

There are exactly eight 6-vertex forbidden induced graphs for the class of mock-threshold graphs. They are two disjoint triangles with 0, 1, 2 or 3 pairwise non-adjacent edges joining the two triangles, and their complements.

Let $\mathcal{M}$ be the set of following graphs:

- Cycles of length at least 5 and their complements
- $K_{3,3}$, domino, K_4 with a matching subdivided, and their complements
- A list of graphs on 7 vertices
- A list of graphs on 8 vertices
- A finite set of split graphs (next slide)

Conjecture

A graph is mock-threshold if and only if does not contain a graph in $\mathcal{M}$ as an induced subgraph.

Even if it goes wrong, we believe at least the following:

Conjecture

There exists a finite set $\mathcal{M}'$ of graphs such that a graph G not containing a hole or antihole of length ≥ 5 is mock-threshold if and only if does not contain a graph in $\mathcal{M}'$ as an induced subgraph.

If true, what would the finiteness really mean?

Mock-threshold and split

Some members of $Forb(\mathcal{G}_k) \cap \mathcal{G}_{Split}$:

G_1: 8-cycle where one side of the bipartition a clique. (self-complementary)
G_2: Path with 9 vertices where the side of the bipartition containing 5 vertices is a clique.
G_3: Disjoint union of three paths, each with 3 vertices, where the side of the bipartition containing 6 vertices is a clique.
G_4: Complement of G_2.
G_5: Complement of G_3.

The absence of G_i, together with the easy fact that split graphs with bounded clique size is a WQO under induced subgraphs should imply that $Forb(\mathcal{G}_k) \cap \mathcal{G}_{Split}$ is finite.

Loss of Well-Quasi-Ordering

A quasi-order is a pair $(Q, \leq)$, where Q is a set and $\leq$ is a reflexive and transitive relation on Q. A quasi-order $(Q, \leq)$ is a **well-quasi-order (WQO)** if for every infinite sequence $q_1, q_2, \ldots$ in $(Q, \leq)$ there exist $i < j$ such that $q_i \leq q_j$.

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Proposition

Threshold graphs are well-quasi-ordered under the induced subgraph relation. Split graphs are well-quasi-ordered under the subgraph relation, but not under the induced subgraph relation. Chordal graphs are not well-quasi-ordered under the subgraph relation.

Proposition

Mock-threshold graphs are not well-quasi-ordered under the subgraph relation.

Mock-threshold and chordal

Threshold $\subset$ *Split* $\subset$ *Chordal* $\subset$ *Weakly Chordal* $\subset$ *Perfect*

Proposition

A mock-threshold graph is chordal if and only if it has an MT-ordering $v_1, \cdots, v_n$ *such that for every* i $(1 \leq i \leq n)$ *such that the degree of* v_i *in* $G : \{v_1, \cdots, v_i\}$ *is* $i - 2$*, the unique non-neighbor of* v_i *in* $G : \{v_1, \cdots, v_i\}$ *is simplicial.*

Questions and Problems

Nested sequence: $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$

1. Easy to design fast recognition algorithms for members in $\mathcal{G}_k$. Can this be used to design faster recognition algorithms for any of their superclasses?
   For instance, how about weakly chordal graphs?
2. Optimization problems for graphs in this class.
   For instance, how about Hamiltonicity? Polynomial-time for $\mathcal{G}_0$ and NP-Complete for $\mathcal{G}_3$. What about $\mathcal{G}_1$ and $\mathcal{G}_2$?
3. What can we say about the Tutte polynomial of threshold graphs? Mock-threshold graphs?
4. Christianson-Reiner Conjecture:
   If G is a connected threshold graph, then $Jac(G) \cong A(G)$.
   How about mock-threshold graphs?
5. Loss of perfection in $\mathcal{G}_2$. Do we at least have χ-boundedness in $\mathcal{G}_k$ for $k \geq 2$?
6. Any connection to more mainstream problems like Gyárfas-Sumner or Erdös-Hajnal?

THANKS FOR YOUR ATTENTION.