Matroids that guarantee their duals as minors

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Motivating question

Which matroids N, when present as minors of larger matroids M, guarantee that their duals are also minors of M?

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Note that if M contains both N and N^* as minors, then M^* also contains both N and N^*.

Obligatory constraints

For matroids M and N, if M contains both N and N^* as minors, then

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We will call these requirements the obligatory constraints.

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We call the matroids that we are looking for dual-present.

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We know $\delta(N) \neq 0$ (N can't contain N^* unless $N \cong N^*$).

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- Assume N is dual-present and $\delta(N) > 0$.
- Construct a matroid M by adding $\delta(N)$ elements to N without increasing the rank.
- This M must have N^* as a minor.
- To obtain N^*, we must contract an independent set of size $\delta(N)$ from M.

By carefully placing the $\delta(N)$ elements, we show that, unless N has a specific structure, M does not have an N^*-minor.

Theorem (T.)

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Proof.

- It is not hard to see that if N is $U_{0,n}$ or $U_{n,n}$, then it is dual-present.

Proof (Cont.)

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- We assume N is dual-present.
- Assume $\delta(N) > 0$ and construct a matroid M by adding $\delta(N)$ elements to N.
- This M satisfies the obligatory constraints and so has an N^*-minor.
- Recall that we must contract an independent set to get N^*.

Proof (Cont.)

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- Contracting gives $L_{N^*} \geq L_N + \delta(N)$ and $C_{N^*} \geq C_N - \delta(N)$.

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- So $C_N \geq L_N + \delta(N)$ and $C_N \leq L_N + \delta(N)$.

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- So $C_N \geq L_N + \delta(N)$ and $C_N \leq L_N + \delta(N)$.
- Thus $C_N - L_N \geq \delta(N)$ and $C_N - L_N \leq \delta(N)$.

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- Construct a new M by adding $\delta(N)$ elements in parallel to an element of K.
- We must contract $\delta(N)$ coloops to get N^*.
- These contractions create no new loops, a contradiction.

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- Construct a new M by adding $\delta(N)$ elements freely to N.
- Then M, and hence N^*, has no coloops.
- So N has no loops and is $U_{n,n}$.

□

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First we consider the 3-connected binary case.

Some relevant matroids

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We let Z_r denote the binary spike of rank r.

$$A_3 = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

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The matrix A_3 is a representation of $Z_3 \cong F_7$.

Theorem (T.)

A 3-connected binary matroid N is dual-present if and only if N is F_7 or F_7^*.

Proof sketch

Using a result of Seymour, it is not hard to show that F_7 and F_7^* are dual-present.

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To prove the converse we show that

- N has a triad and no triangles.
- $\delta(N) = 1$.

Proof sketch (Cont.)

- N has more than one triad.

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- N has no disjoint triads, so N has two triads T_1^* and T_2^* that intersect in a single element x.

We then create a matroid M by adding an element to the intersection of the complementary hyperplanes of T_1^* and T_2^*.

Proof sketch (Cont.)

Using an extension of this sort guarantees that we must contract x to obtain N^* from M.

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We can use this fact to show that $r(N) \leq 5$ (think about restrictions of projective geometries).

Proof sketch (Cont.)

To finish the proof we use the following result

Theorem (Oxley)

Let N be a binary matroid with $r(N) \geq 3$. Then N is 3-connected and has no $\mathcal{W}_4$-minor if and only if $N \cong Z_r$, Z_r^*, or $Z_r \backslash e$ for some $r \geq 3$.

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We use this to show $r(N) \neq 5$

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We use this to show $r(N) \neq 5$ and if $r(N) = 4$, then N is $F_7^* \cong Z_3^*$.

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The proof of this case relies heavily on graph fans. Specifically, fans of maximum size.

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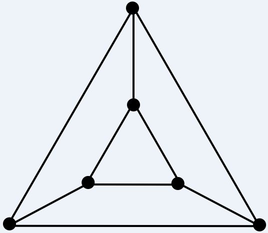

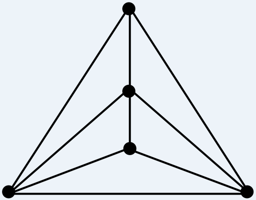

- Note that if N is dual-present and $N \cong M(G)$, then G is planar.

Proof sketch

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- Show G has at least 2 triangular faces and at least 6 vertices of degree 3.

Proof sketch (Cont.)

- Show G has a fan with at least 5 edges.

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- Show a fan of maximum size in G has an odd number of edges, and G has a maximum-sized fan with triads on both ends.

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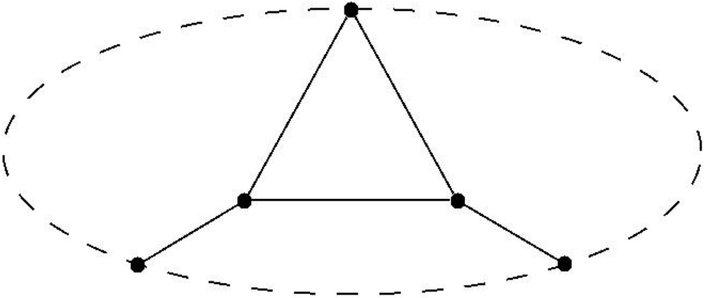
- Show a fan of maximum size in G has an odd number of edges, and G has a maximum-sized fan with triads on both ends.
- Show $\delta(N) = 1$.

Fan exchange



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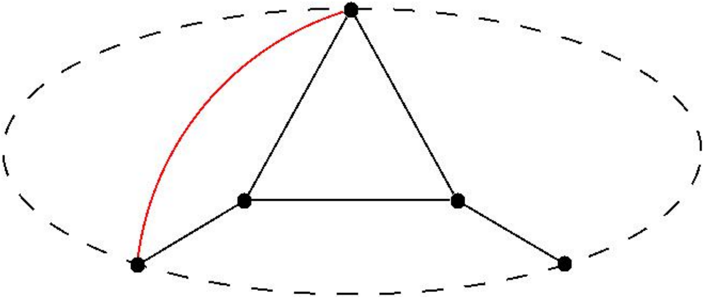

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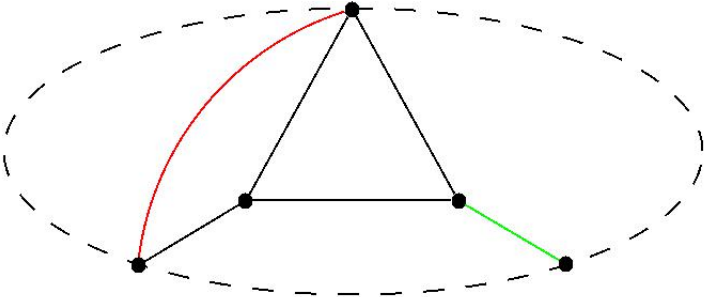

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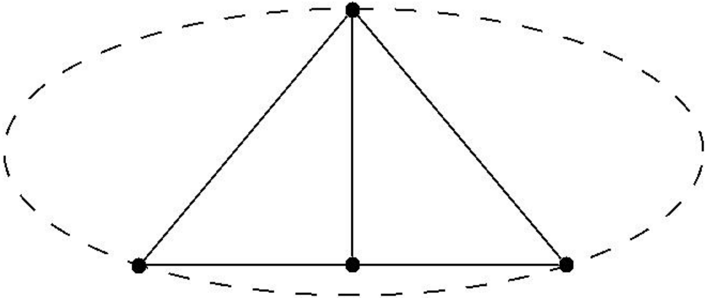

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Proof sketch (Cont.)

- Define t_G = (# of degree-3 vertices) - (# of triangular faces).
- Show $2 \leq t_G \leq 4$.
- Prove $t_G \neq 2$ and $t_G \neq 3$.
- Show if $t_G = 4$, then $G \in \{P, K_5 \backslash e\}$.