On Maximum-Sized Golden-Mean Matroids

Maximum-Sized

Definition

Let $\mathcal{M}$ be a minor-closed class of matroids and let r be a non-negative integer. The ***growth-rate function*** of $\mathcal{M}$ is

$$h_{\mathcal{M}}(r) = \max\left\{\epsilon(M) \mid M \in \mathcal{M} \text{ and } r(M) \leq r\right\},$$

where $\epsilon(M)$ is the number of rank-one flats of M.
We say that M is ***maximum-sized*** in $\mathcal{M}$ if M is a simple matroid in $\mathcal{M}$ such that $\epsilon(M) = h_{\mathcal{M}}(r(M))$.

$GF(3)$-representable matroids

Theorem (Whittle, 1997)

Let $\mathcal{F}$ be a set of fields containing $GF(3)$, and let $\mathcal{M}$ be the class of matroids representable over all fields in $\mathcal{F}$. Then for some $q \in \{2, 3, 4, 5, 7, 8\}$, $\mathcal{M}$ is the class of matroids is the class of matroids representable over $GF(3)$ and $GF(q)$.

Historical Results

Maximum-sized results exist for all of Whittle's $GF(3)$-classes of matroids:

- $GF(q)$-representable matroids for each prime power q, in particular $q = 3$.
- Matroids that are representable over $GF(3)$ and $GF(2)$.
- Matroids that are representable over $GF(3)$ and $GF(5)$.
- Matroids that are representable over $GF(3)$ and $GF(4)$.
- Matroids that are representable over $GF(3)$ and $GF(8)$.
- Matroids that are representable over $GF(3)$ and $GF(7)$.

$GF(q)$-representable matroids

Theorem

The matroid $PG(r-1, q)$ is the maximum-sized matroid of rank-r in the class of $GF(q)$-representable matroids. The growth-rate function of the class of $GF(q)$-representable matroids is $\frac{q^r - 1}{q - 1}$.

Regular matroids

Definition

A matroid is ***regular*** if it is representable over $GF(3)$ and $GF(2)$.

Theorem (Heller, 1957)

The maximum-sized regular matroid of rank r is $M(K_{r+1})$. The growth-rate function of regular matroids is $\binom{r+1}{2}$.

Dyadic matroids

Definition

A matroid is ***dyadic*** if it is representable over $GF(3)$ and $GF(5)$.

Theorem (Kung and Oxley, 1988-90)

The maximum-sized dyadic matroid of rank r is $Q_r(GF(3)^*)$, the rank-r ternary Dowling geometry. The growth-rate function of dyadic matroids is r^2.

Near-regular and $\sqrt[6]{1}$ matroids

Definition

A matroid is ***sixth-root-of-unity*** if it is representable over $GF(3)$ and $GF(4)$. A matroid is ***near-regular*** if it is representable over $GF(3)$ and $GF(8)$.

Theorem (Oxley, Vertigan, and Whittle; 1998)

Except for rank 3, both the maximum-sized rank-r near-regular matroid and the maximum-sized rank-r $\sqrt[6]{1}$ *matroid are* T_r^1*. Both classes have growth-rate function* $\binom{r+2}{2} - 2$, *for* $r > 3$.

$GF(3) \cap GF(7)$-matroids

Theorem

The maximum-sized rank-r matroid representable over $GF(3)$ and $GF(7)$ is $Q_r(GF(3)^*)$. This class has growth-rate function r^2.

$\mathbb{P}$-matrices

Definition

Let $\mathbb{P}$ be a subset of a ring that contains -1 and 0 such that $\mathbb{P}$ is closed under multiplication and division. A $\mathbb{P}$-***matrix*** is a matrix with entries from $\mathbb{P}$ such that every subdeterminant is in $\mathbb{P}$.

Definition

Let A be a $\mathbb{P}$-matrix. A subset of k columns is independent if it contains a $k \times k$ subdeterminant that is non-zero. This gives rise to a $\mathbb{P}$-***representable*** matroid.

Example

The subset $\{-1, 0, 1\}$ of $\mathbb{Z}$ defines the regular matroids.

Definition

The ***golden-mean*** set is $\mathbb{G} = \left\{\pm\tau^i \mid i \in \mathbb{Z}\right\} \cup \{0\}$ where τ is the positive root of $x^2 - x - 1$, also known as the golden ratio.

Golden-mean

Theorem (Semple, Vertigan, Pendavingh, Van Zwam)

Let M be a matroid. The following are equivalent:

(i) *M is representable over both $GF(4)$ and $GF(5)$;*

(ii) *M is golden-mean;*

(iii) *M is representable over $GF(p)$ for all primes p such that $p = 5$ or $p \equiv \pm 1 \pmod 5$, and also over $GF(p^2)$ for all primes p.*

Golden Mean Determinants

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \tau & 1 & 1 & 0 & 0 & \tau & \tau^2 \\ 0 & 0 & 1 & 1 & \tau^2 & 1 & \tau & -\tau & 1 & 1 & \tau^2 \end{bmatrix}$$

- In this matrix, every non-zero subdeterminant is in the set $\{\pm\tau^i \mid i \in \mathbb{Z}\}$.
- For example, $\begin{vmatrix} \tau & \tau \\ \tau^2 & 1 \end{vmatrix} = \tau - \tau^3 = \tau(1 - \tau^2) = \tau(-\tau) = -\tau^2$.

Archer's Conjecture

The conjecture we have been trying to prove is due to Archer.

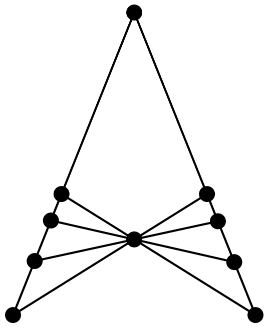
Conjecture

The growth-rate function for the class of golden-mean matroids $\mathcal{G}$ is

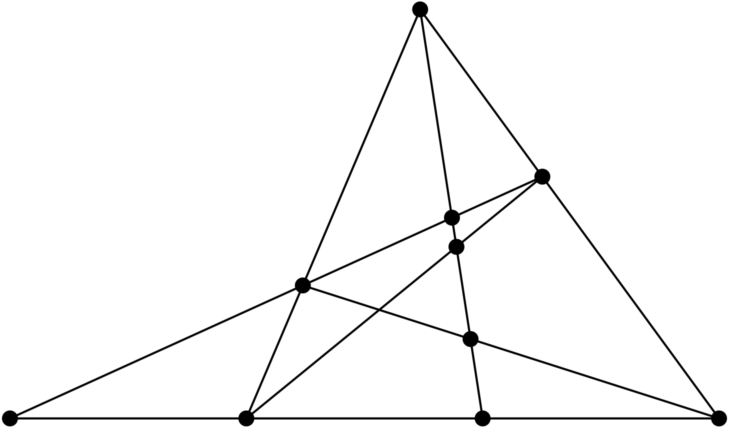
$$h_{\mathcal{G}}(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

Furthermore, $M \in \mathcal{G}$ is maximum-sized if and only if M is isomorphic to a member of $\{T_r^2, G_r, HP_r\}$ when $r(M) \neq 3$, or M is isomorphic to the Betsy Ross when $r(M) = 3$.

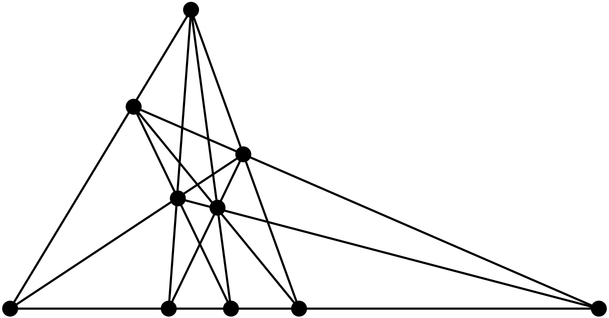
T_3^2



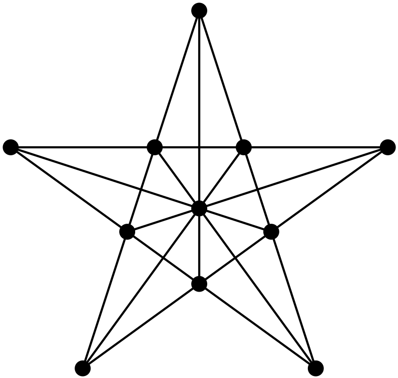
G_3



HP_3



Betsy Ross (B_{11})



The Betsy Ross matroid, or B_{11}.

Spanning Clique

Definition

Let M be a rank-r matroid with a $M(K_{r+1})$-restriction. We call this restriction a ***spanning clique*** for M.

Mayhew-Welsh Theorem

Theorem

Let $\mathcal{M}$ be the set of golden-mean matroids that have spanning cliques. Let $\mathcal{G}$ be the family of minors of matroids in $\mathcal{M}$. Then

$$h_{\mathcal{G}}(r) = \begin{cases} \binom{r+3}{2} - 5 & \text{if } r \neq 3; \\ 11 & \text{if } r = 3. \end{cases}$$

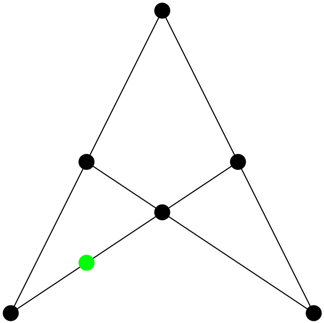
Furthermore, $M \in \mathcal{G}$ is maximum-sized if and only if M is isomorphic to a member of $\{T_r^2, G_r, HP_r\}$ when $r(M) \neq 3$, or M is isomorphic to the Betsy Ross when $r(M) = 3$.

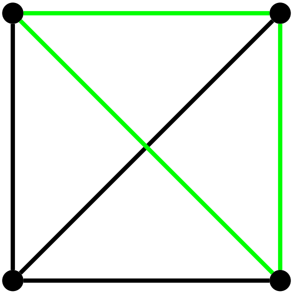
This proves Archer's conjecture for $\mathcal{G}$.

Proof Sketch I

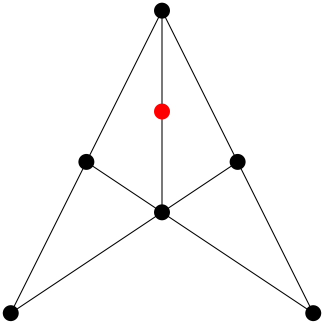
- Step 1: We first show that there are only two ways to extend onto a spanning clique in a golden-mean way. We do this by reducing to a finite case check.
- We call these two ways “green triangles” and “red matchings”.

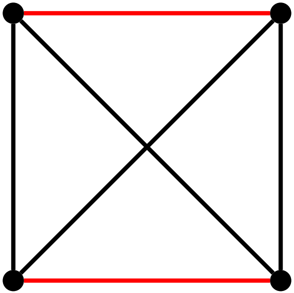
Green Triangles





Red Matchings

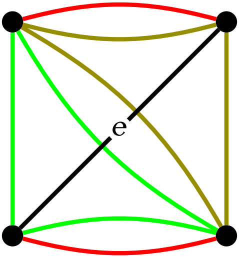


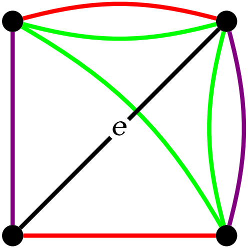


Proof Sketch II

- An ***augmented clique*** is a clique with a multiset of red matchings and green triangles. Every golden-mean matroid with a spanning clique has an associated augmented clique.
- Step 2: We show that in any golden-mean matroid with a spanning clique there must be a clique element that is not in a $U_{2,4}$-restriction. We do this by analysing all ways that a clique element could be in a $U_{2,4}$-restriction.
- There are four such ways.
- The first is a green triangle.
- The second is a doubled-up red matching.

$U_{2,4}$-restrictions





Proof Sketch III

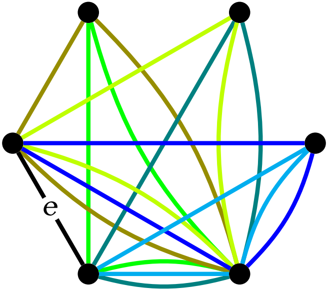
For the rest of the proof, we consider M, a rank-r minimal counterexample to our theorem.

- Step 3: We show that if there exists a clique element e in M that is not in any $U_{2,4}$-restrictions, then e is on at least three non-clique three-point lines.
- This is an induction proof, in the style of the existing maximum-sized results. If e is on k non-clique three-point lines, then $\epsilon(M) - \epsilon(\mathrm{si}(M/e)) \geq r + k$.
- If e is on exactly two non-clique three-point lines, then we know that $\mathrm{si}(M/e)$ is either T^2_{r-1}, G_{r-1}, or HP_{r-1}, and we are able to deduce that M is not a counterexample.

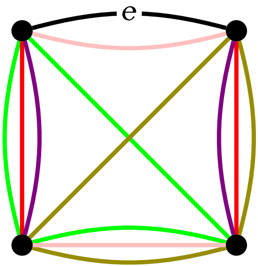
Proof Sketch IV

Step 4: We then analyse all possible ways such an e could arise, and discover that there are four possible configurations.

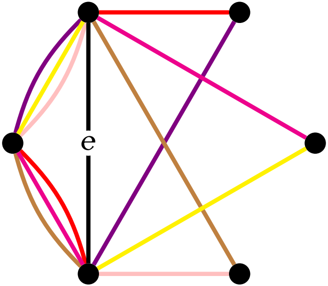
Line-Star Configuration



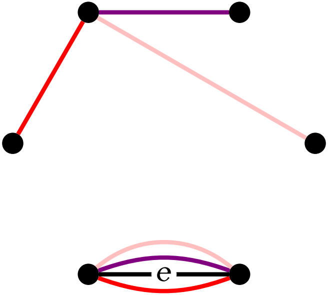
Betsy Ross Configuration



Two-Parallel Configuration



Matching-Star Configuration



Proof Sketch V

Step 5: For each of these four configurations, we show that $\epsilon(M)$ is smaller than $h_{\mathcal{B}}(r(M))$, and hence is not a counterexample.

End Result

It is anticipated that recent work by Geelen and Nelson, when combined with our theorem, will lead to a proof of Archer's Conjecture for matroids with sufficiently large rank.