Rota's Conjecture

Jim Geelen, Bert Gerards, and Geoff Whittle

Rota's Conjecture

For each finite field field $\mathbb{F}$, there are at most a finite number of excluded minors for $\mathbb{F}$-representability.

Ingredients of Proof

1. "Corollaries" of structure theorems for members of proper minor-closed classes of $\mathbb{F}$-representable matroids.
2. Reduction to matroids that are in some sense highly connected.
3. Techniques that generalise those described in Notices paper.

Connectivity Functions

(A, B) partition of $E(M)$. Then

$$\lambda_M(A) = r(A) + r(B) - r(M).$$

If $\lambda_M(A) < k$, and $|A|, |B| \geq k$, then (A, B) is a k-separation of M.

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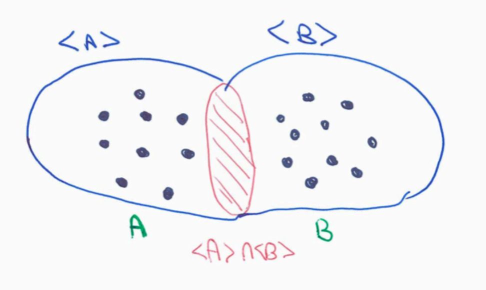
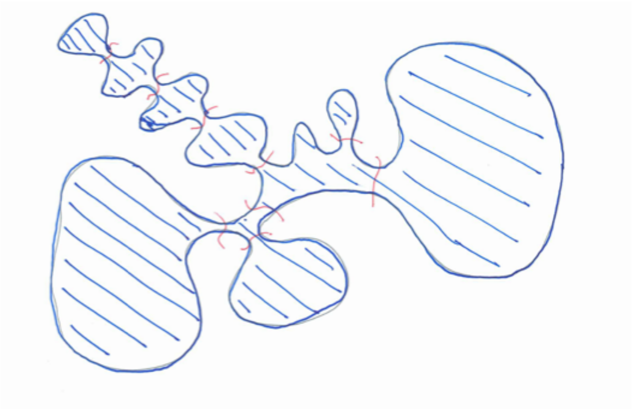
If $\lambda_M(A) < k$, and $|A|, |B| \geq k$, then (A, B) is a k-separation of M.

- If M is represented, then $\lambda_M(A)$ is the rank of $\langle A \rangle \cap \langle B \rangle$.



How can λ behave? Can represent schematically using blob diagrams.



Tutte Connectivity

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Theorem (Notices Paper)

There exists a k such that every k-connected excluded minor has at most $2k$ elements.

f-connectivity

f an integer-valued function. Then M is f-connected if, whenever (A, B) is a k-separation of M, then either $|A| \leq f(k)$ or $|B| \leq f(k)$.

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We prove that excluded minors are f-connected for some f, that is, we prove

Theorem

Let M be an excluded minor for $\mathbb{F}$-representability. Then, for all k there exists n such that, if (A, B) is a k-separation of M, then either $|A| \leq n$ or $|B| \leq n$.

Missing Link

Rota's Conjecture follows from proof that f-connected excluded minors have bounded size. This proof is similar to that presented in the Notices paper using connectivity results of Geelen, Gerards, Huynh, and van Zwam.

Goal of this talk

Give insight into proof that excluded minors are f-connected.

Note

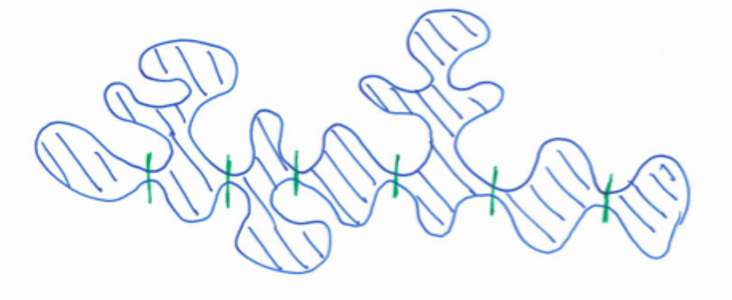
In any unexplained context, $\mathbb{F}$ is a finite field; M is an excluded minor with a k-separation (A, B) where both A and B are extremely large.

Hypotheses

The proof uses unwritten-up theorems from matroid minors project. We'll distinguish these by calling such results Hypotheses.

Controlling the Blobs — Nested k-separations

A sequence of k-separations $(A_1, B_1), (A_2, B_2), \ldots, (A_m, B_m)$ is nested if $A_1 \subset A_2 \subset \cdots \subset A_m$.



Theorem (Ben-David and Geelen)

For each k, there is an n such that no excluded minor admits a sequence of n nested k-separations.

(A, B) a k-separation of a matroid M, where $M|A$ and $M|B$ are both representable. How can we certify that M is representable?

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Local Connectivity

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

In other words,

$$\sqcap(X, Y) = \lambda_{M|(X \cup Y)}(X).$$

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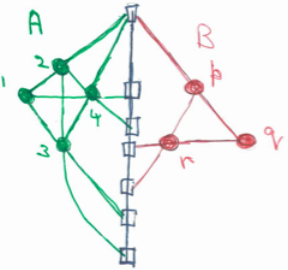
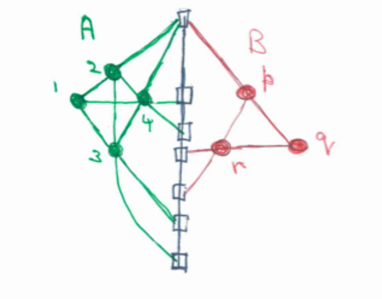
Equivalence

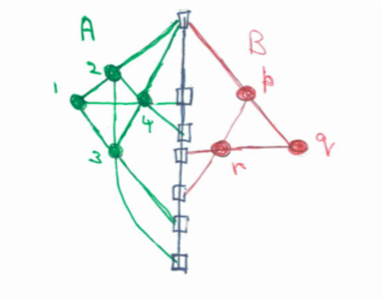
(A, B) a k-separation of M. Then $A_1, A_2 \subseteq A$ are equivalent if $\sqcap(A_1, Z) = \sqcap(A_2, Z)$ for all $Z \subseteq B$.


A
B
1
2
3
4
p
q
r



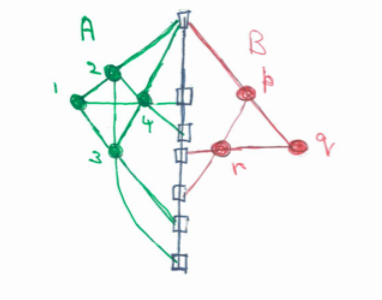
Equivalence classes for (A, B)

$\{1234, 123, 234, 341, 412\},$



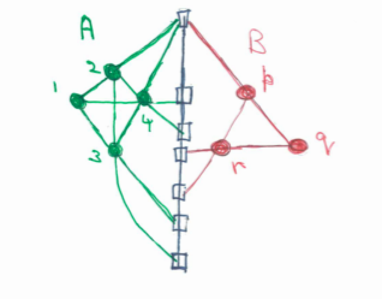
Equivalence classes for (A, B)

$\{1234, 123, 234, 341, 412\}, \{12, 34\},$



Equivalence classes for (A, B)

$\{1234, 123, 234, 341, 412\}, \{12, 34\}, \{14, 23, 13, 24\},$



Equivalence classes for (A, B)

$$\{1234, 123, 234, 341, 412\}, \{12, 34\}, \{14, 23, 13, 24\}, \{\emptyset, 1, 2, 3, 4\}$$

Equivalence classes for (B, A)

$$\{pqr\}, \{pq\}, \{pr, qr\}, \{p, q, r, \emptyset\}$$

(A, B) a k-separation of M. Let $\mathcal{P}(M, A)$ denote the partition of 2^A into equivalence classes.

Lemma

If M is $\mathbb{F}$-representable, (A, B) a k-separation, then $|\mathcal{P}(M, A)|$ is at most the number of flats of $PG(k-2, \mathbb{F})$.

Lemma

If M is an excluded minor for $\mathbb{F}$-representability, then $|\mathcal{P}(M, A)|$ is bounded as a function of k.

$\mathbb{F}$-schemes

(A, B) a k-separation of M. An $\mathbb{F}$-scheme for A is a function σ_A that associates a set of flats of $\mathrm{PG}(k-2, \mathbb{F})$ with each equivalence class of A.

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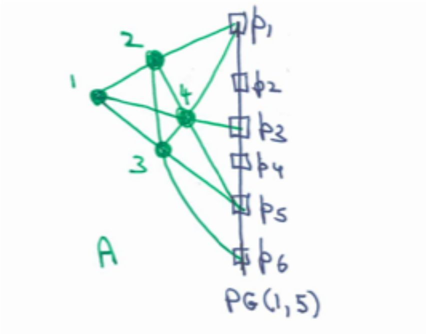
A realizable $\mathbb{F}$-scheme for A is one that can be obtained from a representation of $M|A$.



Realizable $GF(5)$-scheme for A

$\sigma_A(\{1234, \text{etc.}\}) = PG(1,5)$,$\sigma_A(\{12, 34\}) = \{p_1\}$,
$\sigma_A(\{13, 14, 23, 24\}) = \{p_3, p_5, p_6\}$, $\sigma_A\{1, \text{etc.}\} = \emptyset$.

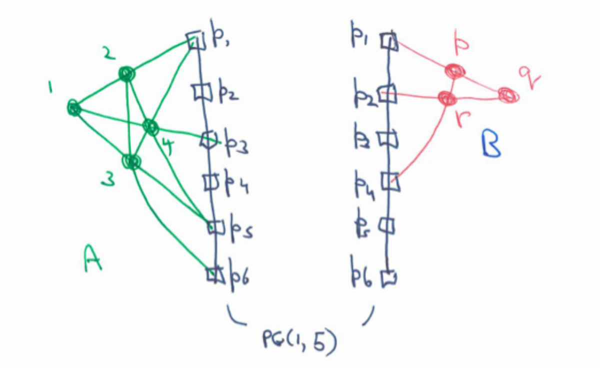
Compatible $\mathbb{F}$-schemes

- Say σ_A and σ_B are $\mathbb{F}$-schemes for (A, B) and (B, A) respectively.
- Say P_A and P_B are equivalence classes of A and B.
- Let $\pi(P_A, P_B)$ denoted the common value of $\sqcap(X_A, X_B)$ for $X_A \in P_A$ and $X_B \in P_B$.
- Then σ_A and σ_B are compatible if, for $F_A \in \sigma(P_A)$ and $F_B \in \sigma(P_B)$, we have $r(F_A \cap F_B) = \pi(P_A, P_B)$.
- Equivalently $\sqcap(F_A \cap F_B) = \pi(P_A, P_B)$.

Lemma

M is $\mathbb{F}$-representable if and only if there exist realizable schemes for (A, B) and (B, A) that are compatible.

Compatible, realizable, GF(5)-schemes for *A* and *B*



Schemes provide the key tool for the proof of Geelen and Ben-David's theorem.

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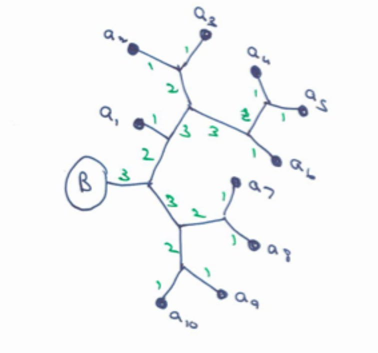
We have a k-separation (A, B) in our excluded minor where both A is large, but we cannot have long nested sequences of t-separations for fixed t. What must we have?

Branch Width of A in M, denoted bw(M, A)

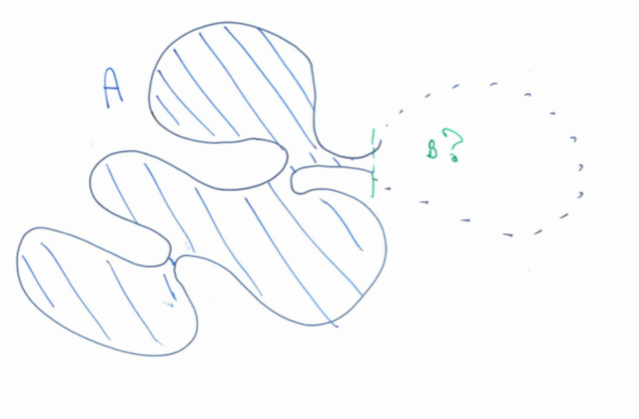


If A is very large and $\mathrm{bw}(M, A) \leq t$, then M has a long nested sequence of t-separations.

Corollary

(A, B) a k-separation in excluded minor M. If A is very large, then A must have large branch width in M.

This means that A must contain big blobs.



Big blobs are identified by high-order tangles.

Tangle Talk

The tangle axioms. A tangle $\mathcal{T}$ of order $\theta + 1$ is a collection of subsets of $E(M)$ such that

- If $X \in \mathcal{T}$, then $\lambda(X) \leq \theta - 1$.
- If (X, Y) is a θ-separation, then either X or Y is in $\mathcal{T}$.
- If $X, Y, Z \in \mathcal{T}$, then $X \cup Y \cup Z \neq E(M)$.
- $E(M) - \{e\} \notin \mathcal{T}$.

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Lemma

If (A, B) is a k-separation of the excluded minor M and A is very large, then A contains a high-order tangle.

Rank in Tangles

Say Z is contained in a $\mathcal{T}$-small set. Then $r_{\mathcal{T}}(Z)$, the rank of Z in $\mathcal{T}$ is defined by

$$r_{\mathcal{T}}(Z) = \min\{\lambda(X) : X \supseteq Z; X \text{ is } \mathcal{T}\text{-small}\}.$$

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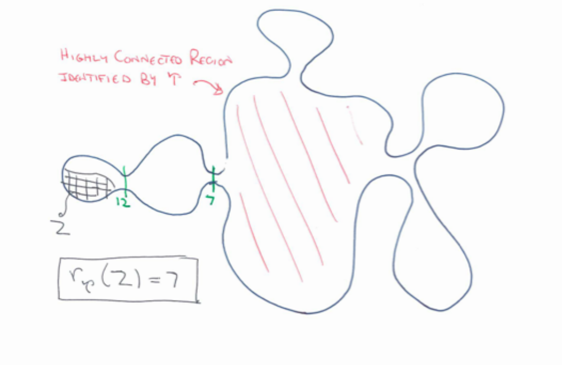
Lemma

$r_{\mathcal{T}}$ is the rank function of a matroid.



Fragile elements

Let M and N be matroids. An element $e \in E(M)$ is N-irrelevant if both $M \backslash e$ and M/e have N as a minor. Otherwise e is N-fragile.

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Note.

Fragility can be difficult to control. A matroid M is N-fragile if it has N as a minor and has no N-irrelevant elements.

- Fragile matroids can be arbitrarily large — whirls are $U_{2,4}$-fragile.
- Indeed, there are $U_{2,4}$-fragile matroids with arbitrarily large branch width.

Hypothesis 1

Let $\mathcal{T}$ be a tangle in the $\mathbb{F}$-representable matroid M; let N be a minor of M and let X be the set of elements of M that are N-fragile. Then there is an integer $n = n(\mathbb{F}, N)$ such that $r_{\mathcal{T}}(X) \leq n$.

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Let $\mathcal{T}$ be a tangle in the $\mathbb{F}$-representable matroid M; let N be a minor of M and let X be the set of elements of M that are N-fragile. Then there is an integer $n = n(\mathbb{F}, N)$ such that $r_{\mathcal{T}}(X) \leq n$.

In Other Words

The set of N-fragile elements has bounded rank in any tangle of M.

Where are we?

- In our counterexample M, both A and B must contain arbitrarily large tangles.
- For any relatively small minor N we can find many elements in A and B that are N-irrelevant.

Applying Hypothesis 1 — keeping equivalence classes.

Keep Equivalence Class Lemma

There exists $b \in B$ such that $\mathcal{P}(M\backslash b, A) = \mathcal{P}(M, A)$.

Lemma

M' an $\mathbb{F}$-representable matroid, (A', B') a k-separation where B' contains a high order tangle $\mathcal{T}$. The set

$$\{b' \in B' : \mathcal{P}(M' \backslash b', A') \neq \mathcal{P}(M', A')$$

has bounded rank in $\mathcal{T}$.

Proof

- Represent M', let $Q = \langle A' \rangle \cap \langle B' \rangle$, and consider matroid N we have on $B' \cup Q$.
- Take all possible minors of N on Q.
- The set of elements that are fragile for each of these minors has bounded rank in $\mathcal{T}$.
- The set of elements that are not irrelevant for all of these minors has bounded rank in $\mathcal{T}$.
- Observe that keeping all these minors guarantees that we keep all equivalence classes.

Keep Equivalence Class Lemma

There exists $b \in B$ such that $\mathcal{P}(M \backslash b, A) = \mathcal{P}(M, A)$.

Proof

- Choose $z \in B$ and consider $M \backslash z$ and M/z.
- Find $b \in B$ that keeps all equivalence classes in both of these matroids.
- Such a b does the job.

Rooted Minors and Well-Quasi-Ordering

- A rooted matroid is a pair (M, S) where $S \subseteq E(M)$.
- minors and isomorphism extend naturally to rooted matroids.

Hypothesis 2

Let S be a finite set and $\mathbb{F}$ be a finite field. In any infinite set of $\mathbb{F}$-representable matroids with root set S, there is one that is a minor of another.

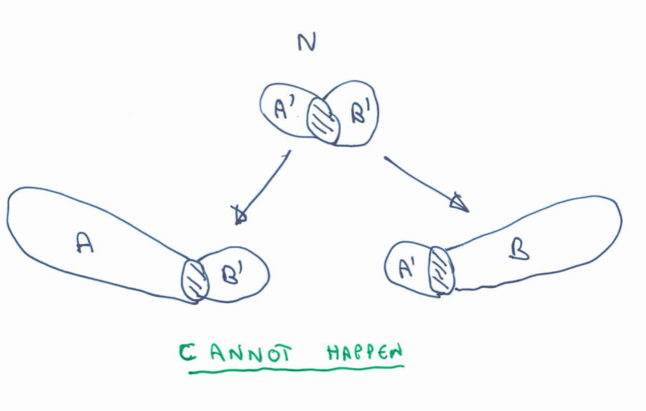
Intertwines

Let $\mathcal{M}$ be a finite set of matroids. A matroid M is an intertwine of $\mathcal{M}$ if M is minor minimal with respect to having every member of $\mathcal{M}$ as a minor.

Corollary of Hypothesis 2

Let $\mathcal{M}$ be a finite set of $\mathbb{F}$-representable matroids. Then there are a finite number of $\mathbb{F}$-representable intertwines of $\mathcal{M}$.

Excluded minor M has "small" minor N such that no representation of N extends both to a representation of the matroid obtained by expanding on the A side and the matroid obtained by expanding on the B side.



Proof

- Consider representation of $M\backslash b$. Let $Q = \langle A \rangle \cap \langle B \rangle$.
- Find minor-minimal matroid N_A that records all possible ways of landing independent sets on Q.
- N_A has bounded size because it is an intertwine.
- Ditto for N_B.
- A bit more work gives N.

Bounding M

- Consider a set $\mathcal{S}$ of representations of N.
- The $\mathbb{F}$ representable matroids that are obtained by extending N on the A'-side and are minor minimal with respect to property that not all representations in $\mathcal{S}$ extend, form an anitchain in the rooted minor order.
- Only finitely many of them.
- Voila!