Matchability of pairs of (possibly infinite) matroids

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2014 International Workshop on Structure
in Graphs and Matroids
July 24, 2014

Matchability

Motivation

Consider a bipartite graph with vertex sides V_M and V_W and the edge set E. Then we have two partition matroids M and W on E, where

- $A \subseteq E$ is independent in M iff no two edges of A have the same endpoint in V_M and
- it is independent in W iff no two edges have the same endpoint in V_W.

Then V_M is matchable into V_W iff there exists a subset $A \subseteq E$ that is spanning in M and independent in W.

Definition (Aharoni and Ziv)

Let M and W be matroids on the same set E. We say that the pair (M, W) is *matchable* iff there exists a subset of E that is spanning in M and independent in W.

μ-admissibility

- A *string* in E is an injective transfinite sequence (function whose domain is an ordinal) of elements of E.
- Given a pair (M, W) of matroids with on E, let μ be a function from the set of all strings in E with values in $\mathbb{Z} \cup \{-\infty, \infty\}$ so that $\mu(f)$ is defined by transfinite induction on $\alpha = \mathrm{dom}(f)$ as follows:
  - $\mu(f) = 0$ if $\alpha = 0$.
  - $\mu(f) = \mu(f \restriction \beta) + \varepsilon_\beta$ if $\alpha = \beta + 1$, where:
    - $\varepsilon_\beta = 1$ if $f(\beta)$ is not spanned by $\mathrm{ran}(f \restriction \beta)$ in both M^* and W;
    - $\varepsilon_\beta = -1$ if $f(\beta)$ is spanned by $\mathrm{ran}(f \restriction \beta)$ in both M^* and W;
    - $\varepsilon_\beta = 0$ otherwise.
  - $\mu(f) = \liminf \{\mu(f \restriction \beta) : \beta < \alpha\}$ if α is a limit ordinal.
- The pair (M, W) is μ-admissible iff $\mu(f) \geq 0$ for every string f in E.

Finite bipartite graphs and μ-admissibility

Observation

Let M and W be the partition matroids obtained from a finite bipartite graph with sides V_M and V_W. If f is any string in E, then

$$\mu(f) = i_W - c_M,$$

where

- i_W is the number of vertices v in V_W for which there exists an edge incident to v that is in $\mathrm{ran}(f)$ and
- c_M is the number of vertices v in V_M for which all edges incident to v are in $\mathrm{ran}(f)$.

Hall's Theorem

If M and W are the partition matroids obtained from a finite bipartite graph, then the pair (M, W) is matchable iff it is μ-admissible.

Matchability $\Rightarrow$ μ-admissibility

Theorem

Any matchable pair (M, W) of matroids is μ-admissible.

Proof.

Let $T \subseteq E$ be spanning in M and independent in W and let $S = E \setminus T$. If f is a string in E, then let

$$T_f = T \cap \text{ran}(f) \text{ and } S_f = S \cap \text{ran}(f).$$

We prove by transfinite induction on $\alpha = \text{dom}(f)$ that

$$\mu(f) \geq r((M^*/S_f) \,|\, T_f) + r((W/T_f) \,|S_f),$$

where r is the rank function (with values in $\mathbb{N} \cup \{\infty\}$). Since the values of r are nonnegative, it follows that $\mu(f) \geq 0$. $\square$

Bipartite graphs

Theorem (Nash-Williams)

Let (M, W) be the pair of matroids obtained from a bipartite graph such that every vertex in V_W has countable degree. If (M, W) is μ-admissible, then it is matchable.

Proof.

Let f be a surjective string in V_M and $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any bijection.

> *Lemma: For every $m \in V_M$ there exists $w \in V_W$ adjacent to m such that the pair of matroids obtained from the graph with m and w removed is still μ-admissible.*

Inductively, go along the string f using the Lemma. After obtaining suitable $w \in V_W$, list all (countably many) $m \in V_M$ that are adjacent to w. Using φ process all m from those new lists before arriving at a limit step. Then the pair of matroids obtained from the remaining graph is still μ-admissible. □

Bipartite graphs

Counterexample

Consider a complete bipartite graph with V_M uncountable and V_W countably infinite. Then the corresponding pair (M, W) of matroids is not matchable.

Fact

The pair (M, W) is μ-admissible.

Proof.

Suppose, for a contradiction, that f is a string in E with $\mu(f) < 0$. Let $\beta < \text{dom}(f)$ be the smallest ordinal such that $f(\beta)$ is spanned by $\text{ran}(f \restriction \beta)$ in M^*. Let $m \in V_M$ be incident to $f(\beta)$. Then all the other edges incident to m must appear in $f \restriction \beta$. For each of those edges take the first edge in f that is adjacent to the same vertex in V_W. None of the obtained edges is spanned by the preceding edges in either M^* or W, implying that $\mu(f \restriction \beta) = \infty$. □

Countable finitary counterexample

Theorem (Aharoni, Thomassen)

For each integer $k \geq 3$ there exists a countable graph G such that G is $2k$-edge connected but has no k edge-disjoint spanning trees.

Counterexample

Let M be the cycle matroid of the graph consisiting of disjoint k copies of G and W be the partition matroid on the same ground set with each part consisting of all copies of the same edge of G.

Fact

The pair (M, W) is not matchable.

Theorem (Bruhn)

The pair (M, W) is μ-admissible.

When μ-admissibility implies matchability?

Theorem

Let E be countable, M be cofinitary and W be finitary. If (M, W) is μ-admissible, then it is matchable.

Theorem

Let M be a countable sum of finite-rank matroids and W be finitary. If (M, W) is μ-admissible, then it is matchable.

Remark

In the theorems above and in the result concerning bipartite graphs, for every $e \in E$ the numbers of cocircuits through e in M and circuits through e in W are both countable.

Conjecture

Let M and W be such that for each $e \in E$ there are countably many cocircuits in M through e and countably many circuits in W through e. If (M, W) is μ-admissible, then it is matchable.

General definition of a matroid

Matroid = Higgs' B-matroid

Independence Axioms

(I1) $\varnothing$ is independent;

(I2) a subset of an independent set is independent;

(I3) if A is a maximal independent set and B is independent but not maximal, then there is $a \in A \smallsetminus B$ such that $B \cup \{a\}$ is independent;

(IM) if $A \subseteq B$ and A is independent, then the set

$$\{X : A \subseteq X \subseteq B \text{ and } X \text{is independent}\}$$

contains a maximal element.

Theorem

Let M be a countable sum of finite-rank matroids and W be finitary. If (M, W) is μ-admissible, then it is matchable.

Proof.

Assume $M = M' \oplus M''$ with M' having positive finite rank. We show that there exists a maximal string f in E such that $\mu(f) = 0$ and for some $a \in E \setminus \operatorname{ran}(f)$ the singleton $\{a\}$ is independent in M'. We construct two pairs (M_1, W_1) and (M_2, W_2) of μ-admissible matroids as follows:

- M_1^* and W_1 are obtained by contracting $\{a\}$ in both M^* and W and then restrict the resulting matroids to $\operatorname{ran}(f)$;
- to get M_2^* and W_2 contract $\operatorname{ran}(f) \cup \{a\}$ in M^* and W.

Repeat the construction for both pairs as long as the rank of the corresponding M_i is nonzero. The set A of all the elements a selected at each step is then spanning in M. After each step the finite part of A selected so far is independent in W. Since W is finitary, A is independent in W. □