The 3-Connected Binary Matroids with no P_9-minor

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Some well-known excluded-minor results

Kuratowski's Theorem

A graph is planar if and only if it has no minor that is isomorphic to $K_{3,3}$ or K_5.

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H-free

Let M and H be matroids. We say that M is H-free if M has no minor isomorphic to H.

Some well-known excluded-minor results

Theorem (Tutte, 1965)

A matroid M is binary if and only if M has no $U_{2,4}$-minor.

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Figure: $U_{2,4}$

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Figure: $U_{2,4}$

Theorem (Tutte, 1965)

A matroid is regular if and and only if M has no $U_{2,4}$, F_7 (Fano) or F_7^*-minor.

Notations

The Fano Matroid F_7

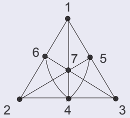

Figure: The Fano Matroid

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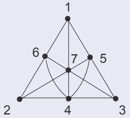

Figure: The Fano Matroid

$EX_G(G_1, G_2, \ldots, G_k)$: the set of 3-connected graphs with no G_1, G_2, ..., or G_k-minor.
$EX(M_1, M_2, \ldots, M_k)$: the set of 3-connected binary matroids with no M_1, M_2, ..., or M_k-minor.

Some definitions

Weak Splitter

Let $\mathcal{N}$ be a minor-closed class of matroids and N be a 3-connected member of $\mathcal{N}$. We say that N is a *weak splitter* of $\mathcal{N}$, if, there is a finite set of 3-connected matroids $\mathcal{T} \subseteq \mathcal{N}$, such that for any 3-connected matroid M of $\mathcal{N}$, if M has an N-minor, then $M \in \mathcal{T}$.

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Internally 4-connected Matroids

A matroid M is internally 4-connected if it is 3-connected and for any 3-separation (X, Y) of M, X (or Y) is a triangle or a triad.

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Weakly 4-connected Matroids

A matroid M is weakly 4-connected if it is 3-connected and for any 3-separation (X, Y) of M, either $|X| \leq 4$, or $|Y| \leq 4$.

3-Sum of two Fano matroids

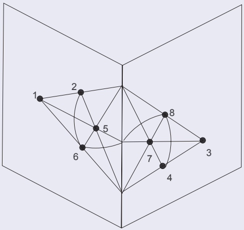

Figure: AG(3, 2): the 3-sum of two Fano Matroids

3-Sum of two Fano matroids

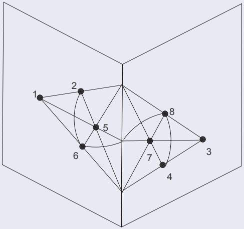

Figure: AG(3, 2): the 3-sum of two Fano Matroids

3-Sum of two prism graphs

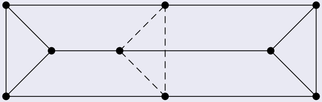

Figure: A 3-sum of two prisms

Small 3-connected graphs

Notation

Let $\mathcal{K}$ denote the class of simple 3-connected graphs G for which can be obtained from $K_{3,n}$ $(n \geq 1)$ by adding edges to its color class of size three. Let $\mathcal{W} = \{W_n : n \geq 3\}$. We use Prism to denote both the prism graph and its cycle matroid.

Simple 3-connected graphs with at most 11 edges

$\lvert E(G)\rvert$	simple 3-connected graphs
at most 8	W_3, W_4
9	$K_{3,3}, K_5\backslash e$, Prism
10	$W_5, K_5, K_{3,3}+e$, Prism+e
11	7 graphs

Excluding small 3-connected graphs

Theorem (Hall, 1943)

$EX_G(K_{3,3})$ consists of K_5 and 3-connected planar graphs.

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Theorem (Dirac 1963, Lovasz 1965)

$EX_G(Prism) = \{K_5\} \cup \mathcal{W} \cup \mathcal{K}$.

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Corollary

$EX_G(K_{3,3}+e) = \{K_{3,3}, K_5\} \cup \mathcal{P}$.

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Theorem

$EX_G(Prism+e) = \{Prism, K_5\} \cup \mathcal{P} \cup \mathcal{W}$.

Excluding small 3-connected graphs

Theorem (Wagner, 1937)

$EX_G(K_5)$ consists of V_8 and (3-connected members of) 3-sums of 3-connected planar graphs.

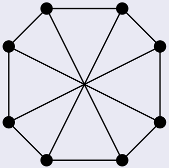

Figure: The graph V_8

Excluding small 3-connected graphs

Theorem (Oxley, 1989)

Let G be a graph. Then G is simple, 3-connected, and W_5-free if and only if $G \cong \{W_3, W_4\}$, $G \in \mathcal{K}$ or G is a 3-connected minor of the cube, octahedron, pyramid, or $K_5^{\perp}$.

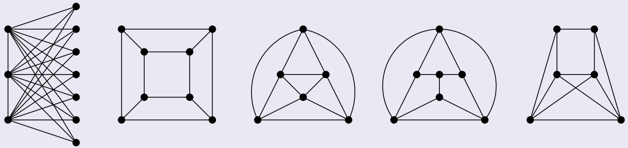

Figure: The graphs $K''_{3,7}$, cube, octahedron, pyramid, and $K_5^{\perp}$

Excluding small 3-connected graphs

Remark

Ding and Liu (2013) characterized the 3-connected H-free graphs for any 3-connected graph H with eleven edges.

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V_8-free

Robertson characterized the 3-connected V_8-free graphs.

Cube-free

Maharry (2000) characterized the 3-connected cube-free graphs.

Theorem (Ding, 2013)

A graph G is octahedron-free if and only if it is constructed by 0-, 1-, 2-, and 3-sums starting from graphs in the set $\{K_1, K_2, K_3, K_4\} \cup \{C^2_{2n+1} : n \geq 2\} \cup \{L'_4, L_5, P_{10}, L'_5, L''_5, \}$.

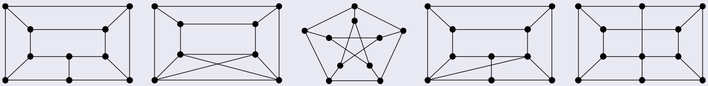

Figure: The graphs $L_5, L'_4, P_{10}, L'_5, L''_5$

Excluding small 3-connected binary matroids

Small 3-connected binary matroids

$\lvert E(M) \rvert$	Binary 3-connected matroids
6	W_3
7	F_7, F_7^*
8	$W_4, S_8, AG(3,2)$
9	$M(K_{3,3}), M^*(K_{3,3}), M(K_5 \backslash e), Prism, P_9, P_9^*, Z_4, Z_4^*$

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P_9

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Figure: A binary representation for P_9

Excluding small 3-connected binary matroids

All 3-connected binary matroids have an $M(K_4)$-minor

$EX(M(K_4)) = \emptyset$.

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Theorem (Seymour, 1980)

$EX(F_7) = \{F_7^*\} \cup$ {the set of 3-connected regular matroids}
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Proposition (S_8-free)

$EX(S_8)=\{F_7, F_7^*, AG(3,2)\}$.

Excluding small 3-connected binary matroids

Theorem (Oxley, 1987)

$EX(M(W_4))=\{Z_r, Z_r^*, Z_r\backslash b_r, \text{ or } Z_r\backslash c_r\}$ $(r \geq 3)$ (plus some trivial matroids with at most three elements).

Excluding small 3-connected binary matroids

Theorem (Oxley, 1987)

$EX(M(W_4))=\{Z_r, Z_r^*, Z_r\backslash b_r, \text{ or } Z_r\backslash c_r\}$ $(r \geq 3)$ (plus some trivial matroids with at most three elements).

Remark

These matroids are called binary spikes.

Excluding small 3-connected binary matroids

Theorem (Mayhew, Royle, and Whittle, 2010)

An internally 4-connected binary matroid M is $M(K_{3,3})$-free if and only if M is either
(i) cographic;
(ii) isomorphic to a member of two infinite classes of binary matroids called trianglar or triadic Möbius matroids; or
(iii) isomorphic to one of eighteen sporadic matroids.

Excluding small 3-connected binary matroids

Theorem (Mayhew and Royle, 2012; Kingan and Lemos 2014)

An internally 4-connected binary matroid M is prism-free if and only if M is isomorphic to one of forty-two matroids, and each is a minor of a 17-element matroid.

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Corollary (Mayhew and Royle, 2012)

Let M be a 3-connected binary Prism-free matroid. Then one of the following is true:
(i) M is one of the 42 internally 4-connected minors of P_{17}; or
(ii) M is one of five sporadic matroids; or
(iii) M can be constructed from copies of $M(K_4)$ and F_7 using parallel extensions and 3-sums.

Theorem (Harville, Reid, and Wu, 2014)

An internally 4-connected binary matroid M is (prism+e)-free if and only if M is isomorphic to one of ninety matroids. Each such matroid has at most seventeen elements, and each is a minor of one of five matroids.

Theorem (Harville, Reid, and Wu, 2014)

An internally 4-connected binary matroid M is (prism+e)-free if and only if M is isomorphic to one of ninety matroids. Each such matroid has at most seventeen elements, and each is a minor of one of five matroids.

Remark

Characterizing internally 4-connected binary $AG(3,2)$-free matroids is still open.

Excluding small 3-connected binary matroids

Theorem (Oxley, 1987)

Let M be binary matroid. The M is 3-connected having no minor isomorphic to P_9 or P_9^* if and only if
(i) M is regular and 3-connected;
(ii) M is a binary spike $Z_r, Z_r^*, Z_r\backslash b_r$ or $Z_r\backslash c_r$ for some $r \geq 4$; or
(iii) $M \cong F_7$ or F_7^*.

Question

Characterize all 3-connected binary matroids with no P_9-minor.

Weaky 4-connected binary P_9-free matroids

Theorem 1 (Ding and Wu)

A binary matroid M having at least four elements is weakly 4-connected and P_9-free if and only if
(i) M is weakly 4-connected graphic or cographic; or
(ii) M is isomorphic to R_{10}; or
(iii) M is one of the 25 weakly 4-connected non-regular sporatic matroids.

The matroid Y_{16}

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$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure: A binary standard representation for Y_{16}

Main Results

Corollary

A binary matroid M having at least four elements is internally 4-connected and P_9-free if and only if
(i) M is internally 4-connected graphic or cographic; or
(ii) M is isomorphic to R_{10}; or
(iii) M is one of the 16 internally 4-connected non-regular minors of Y_{16}.

Corollary

Let M be a 3-connected binary P_9-free matroid. Then one of the following is true:
(i) M is one of the 16 internally 4-connected non-regular minors of Y_{16}; or
(ii) M is regular; or
(iii) M can be constructed from copies of F_7, graphic, or cographic internally 4-connected matroids using parallel extensions and 3-sums.

A new class of binary matroids

Multi-legged starfish

Start from 3-connected matroids $M = M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$. Take any t disjoint triangles $T_1, T_2, \ldots, T_t$ $(1 \leq t \leq n)$ and t copies of F_7. Perform 3-sum operations consecutively starting from M and F_7 along the triangles T_i $(1 \leq i \leq t)$. The resulting matroid is called a multi-legged starfish.

Main Result

Theorem 2 (Ding and Wu)

Let M be a binary matroid. Then M is 3-connected having no minor isomorphic to P_9 if and only if one of the following is true:
(i) M is one of the 16 internally 4-connected non-regular minors of Y_{16}; or
(ii) M is regular and 3-connected; or
(iii) M is a binary spike with rank at least four; or
(iv) M is a multi-legged starfish.

An Outline of Proof for Theorem 2

Step 1

Find all weakly and internally 4-connected binary matroids with no P_9-minor.

X_{10}

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure: A binary representation for X_{10}

X_{10}

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Figure: A binary representation for X_{10}

X_{10} is internally 4-connected, non-regular, and has an $M(K_{3,3})$-minor.

Finding all 3-connected binary P_9-free matroids with an X_{10}-minor

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There are exactly fourteen 3-connected binary P_9-free matroids having an X_{10}-minor; each such matroid is internally 4-connected.

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X_{10} is a weak splitter for the class of binary matroids with no P_9-minor.

There are exactly fourteen 3-connected binary P_9-free matroids having an X_{10}-minor; each such matroid is internally 4-connected.

Each matroid is a non-regular minor of Y_{16}.

$\{P_9, X_{10}\}$-free internally 4-connected and weakly 4-connected binary matroids

Theorem (Zhou, 2012)

Let N be an internally 4-connected minor of a weakly 4-connected binary matroid M with $|E(N)| \geq 7$. For $(N', M') \in (N, M), (N^*, M^*)$, suppose that

- if $N'{=}D_n$, then D_{n+1} is not a minor of M'; and
- if $N' = D_n \backslash f_1$, then $D_{n+1} \backslash f_1$ is not a minor of M'; and
- if $N' = D^n$, then D^{n+1} is not a minor of M'; and
- if $N' = D^n \backslash f_1$, then $D^{n+1} \backslash f_1$ is not a minor of M'.

Then there exists a sequence $M_0, M_1, ..., M_k$ of weakly 4-connected matroids such that $M_0 = N, M_k = M$, and for each $\in \{1, 2, ..., k\}$, M_{i-1} is a proper minor of M_i and $|E(M_{i-1}) \geq |E(M_i)| - 2$.

An Outline of Proof for Theorem 3

Sage computations for internally 4-connected or weakly 4-connected binary matroids with no $\{P_9, X_{10}\}$-minor

$\lvert E(M)\rvert$	i4c	Weakly 4-connected	Matroids
8	0	2	$S_8, AG(3,2)$
9	0	3	A_9, B_9, C_9
10	0	2	A_{10}, B_{10}
11	0	1	A_{11}
12	0	1	A_{12}
13	0	0	
14	0	0	

Step 2

Let M be a 3-connected but internally 4-connected binary P_9-free matroid. Suppose that $M = M_1 \oplus_3 M_2$.
(i) If M_2 is graphic, then $M_2 \cong Prism$, or $M(W_4)$, or $si(M_2) \cong M(W_3)$; and
(ii) If M_2 is cographic but not graphic, then $M_2 \cong M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 3)$.

Step 3

Let M be a 3-connected but not internally 4-connected binary P_9-free matroid. Suppose that $M = M_1 \oplus_3 M_2$. If M_2 is regular, then $M_2 \cong Prism$, $M(W_4)$, $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 3)$, or $si(M_2) \cong M(W_3)$.

Step 3

Let M be a 3-connected but not internally 4-connected binary P_9-free matroid. Suppose that $M = M_1 \oplus_3 M_2$. If M_2 is regular, then $M_2 \cong Prism$, $M(W_4)$, $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 3)$, or $si(M_2) \cong M(W_3)$.

Step 4

Characterize all 3-connected binary P_9-free matroids.

Thank You