Counting matroids in minor-closed classes

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Motivation

Asymptotic proportion of matroids with property $\mathcal{P}$:

$$\lim_{n\to\infty} \frac{\#\left\{M \in \mathbb{M}_n : \ M \text{ has property } \mathcal{P}\right\}}{\#\mathbb{M}_n}$$

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Even the most elementary questions about the properties of "almost all" matroids are currently unanswered.

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Almost all matroids...

- ... contain no (co-)loop (Mayhew, Newman, Welsh, Whittle (2011))
- ... are 3-connected (Lowrance, Oxley, Semple, Welsh (2013))

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Conjecture (Mayhew, Newman, Welsh, Whittle (2011))

N sparse paving $\Rightarrow$ almost every matroid has an N-minor.

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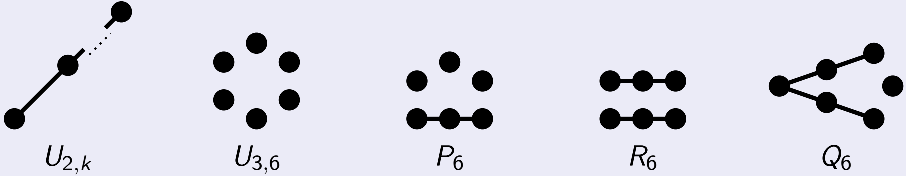
The conjecture holds for $N =$



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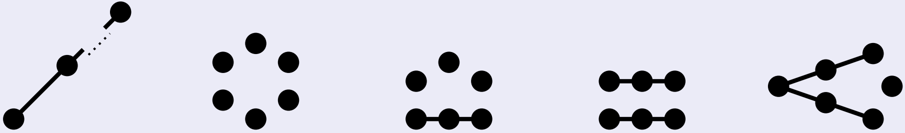
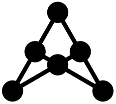
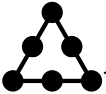


$U_{2,k}$ $U_{3,6}$ P_6 R_6 Q_6

Proof idea: Show that matroids in $\mathrm{Ex}(N)$ have a concise description.

What about , ? Stay tuned!

Cover complexity

X is dependent in $M \iff$ there is a flat F such that $|X \cap F| > r_M(F)$

Definition (Flat cover)

$\mathcal{Z} \subseteq \mathcal{F}(M)$ such that each non-basis is covered by some $F \in \mathcal{Z}$.

- Example: $\mathcal{Z} =$ all hyperplanes

Definition (Cover complexity)

$\kappa(M) =$ minimum size of a flat cover of M.

Properties

Dual-invariant

$\kappa(M) = \kappa(M^*)$

Minor-monotone

If $N \preccurlyeq M$, then $\kappa(N) \leq \kappa(M)$

"Subadditive"

$\kappa(M) \leq \kappa(M/e) + \kappa(M \backslash e)$
(e neither a loop nor a coloop)

Decreasing under relaxing circuit-hyperplanes

$\kappa(M) = \kappa(N) + 1$
(N obtained by relaxing circuit-hyperplane)

Fractional cover complexity

Cover complexity

$$\kappa(M) = \min \left\{ \sum_F z(F) \;\middle|\; \begin{array}{ll} \sum\limits_{F:F \text{ covers } X} z(F) \geq 1 & \forall X \text{ non-basis} \\ z(F) \in \{0,1\} & \forall F \end{array} \right\}$$

Fractional cover complexity = linear relaxation

$$\kappa^*(M) = \min \left\{ \sum_F z(F) \;\middle|\; \begin{array}{ll} \sum\limits_{F:F \text{ covers } X} z(F) \geq 1 & \forall X \text{ non-basis} \\ z(F) \geq 0 & \forall F \end{array} \right\}$$

By randomised rounding:

$$\kappa(M) \leq \kappa^*(M) \left\lceil \ln \frac{\binom{n}{r}}{\kappa^*(M)} + 1 \right\rceil$$

Programme

Theorem (Main technical result)

Let $\mathcal{M}$ be a contraction-closed class. If for some $s \in \mathbb{N}$ and $c, \varepsilon > 0$

$$\max\{\kappa(M) : M \in \mathcal{M} \cap \mathbb{M}_{n,s}\} \leq cn^{s-1-\varepsilon}$$

then

$$\log |\mathcal{M} \cap \mathbb{M}_n| \leq O\left(\frac{1}{n^{3/2+\varepsilon}} 2^n \log^2 n\right).$$

Compare to Knuth's lower bound: $\log |\mathbb{M}_n| \geq \Omega\left(\frac{1}{n^{3/2}} 2^n\right)$.

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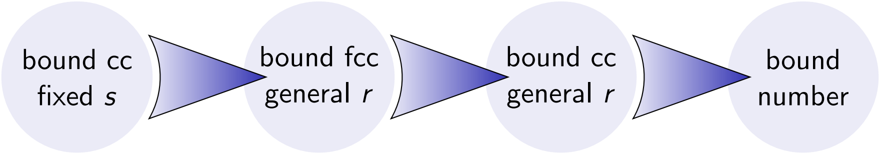

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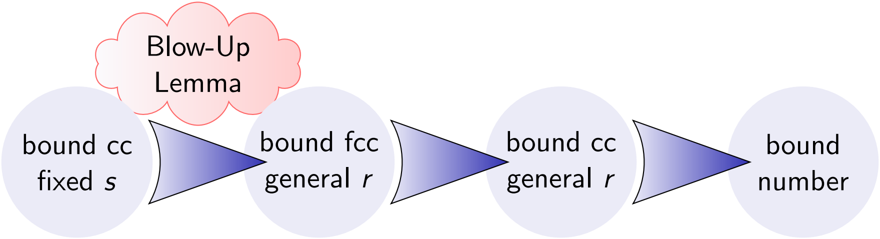

Blow-Up Lemma (1/2)

Lemma (Blow-Up Lemma)

For all $M \in \mathbb{M}_{n,r}$ *and* $t < r$

$$\frac{1}{\binom{n}{r}}\kappa^*(M) \leq \frac{1}{\binom{n-t}{r-t}} \max\left\{\kappa^*(M/S) : S \in \binom{E}{t}\right\}$$

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- Put $z(F) = \frac{1}{\binom{r}{t}} \sum_S z^S(F)$

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S dependent

$z^S = \{\mathrm{cl}\,(S)\}$

Since
$|\mathrm{cl}\,(S) \cap X| \geq |S \cap X| > r(\mathrm{cl}\,(S))$

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$z^S(F) = z_{M/S}(F - S), \quad S \subseteq F$

Since
If F' covers $X - S$ in M/S, then
$F' \cup S$ covers X in M

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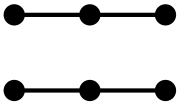
Bound cost:

$$\kappa^*(M) \leq \sum_F z(F) = \frac{1}{\binom{r}{t}} \sum_S \sum_F z^S(F).$$

Claim: $\sum_F z^S(F) \leq \max\left\{\kappa^*(M/S) : S \in \binom{E}{t}\right\}$

Application: Ex(R_6) is small (1/2)



Theorem (Main technical result; repeated)

Let $\mathcal{M}$ be a contraction-closed class. If for some $s \in \mathbb{N}$ and $c, \varepsilon > 0$

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- Apply with $\mathcal{M} = \mathrm{Ex}(R_6)$, $s = 3$.

Lemma

Let $M \in \mathbb{M}_{n,3}$, such that $si(M)$ has L long lines (≥ 3 elts), then $\kappa(M) \le 1 + n/2 + L$.

Application: Ex(R_6) is small (2/2)

Lemma

Let $M \in Ex(R_6) \cap \mathbb{M}_{n,3}$, then $si(M)$ contains at most $3n/2$ long lines.

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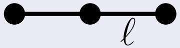

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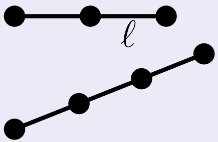

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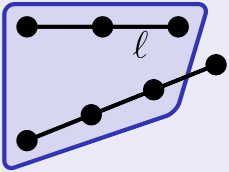

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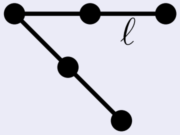

- Any other long line intersects ℓ
- At most $3(n-3)/2$ other long lines

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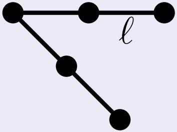

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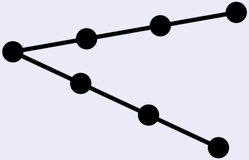

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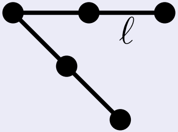

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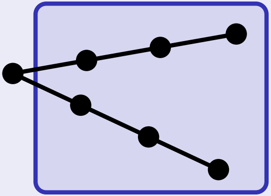

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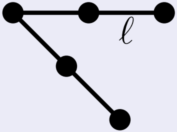

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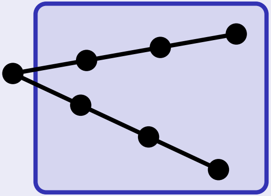
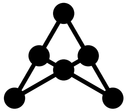


- At most 1 long line

The problem with Ex $(M(K_4))$



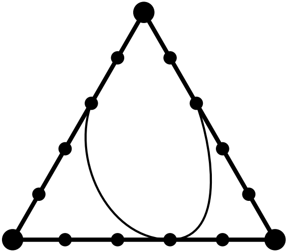
Lemma

Best lower bound on $\#\mathrm{Ex}(M(K_4)) \cap \mathbb{M}_n$
is asymptotic to
best lower bound on $\mathbb{M}_n$.

- Best lower bound on $\#\mathbb{M}_n$: explicit construction of large family of matroids.
- Almost all matroids constructed in this way are in Ex $(M(K_4))$.

The problem with $\mathrm{Ex}(U_{3,7})$

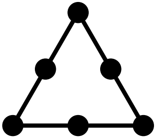
There are matroids in $\mathrm{Ex}(U_{3,7})$ with large cover complexity



Rank-s Dowling matroid $Q_s(\mathrm{GF}(q)^\times)$

- has $s + (q-1)\binom{s}{2} = \Theta(q)$ elements
- has cover complexity $\geq (q-1)^{s-1}$
- has no $U_{3,7}$-minor

Open problem: Cover complexity of Ex (W^3)



Definition

A cap-set is a set of points in AG$(3, m)$ that does not contain a full line. Maximum size: c_m.

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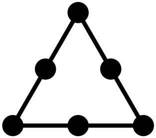

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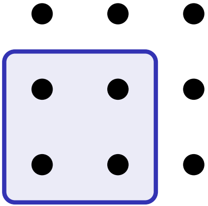

- $c_m \geq 2^m$
- $c_m \geq (2.2174\ldots)^m$ (Edel (2004))
- $c_m \leq O(3^m/m^{1+\delta})$ (Bateman, Katz (2012))

Open problem: Cover complexity of Ex $\left(W^3\right)$

Lemma (Blokhuis)

There exists $M \in Ex\left(W^3\right)$ of rank 3 on 3^{m+1} elements, containing $3^m c_m$ long lines.

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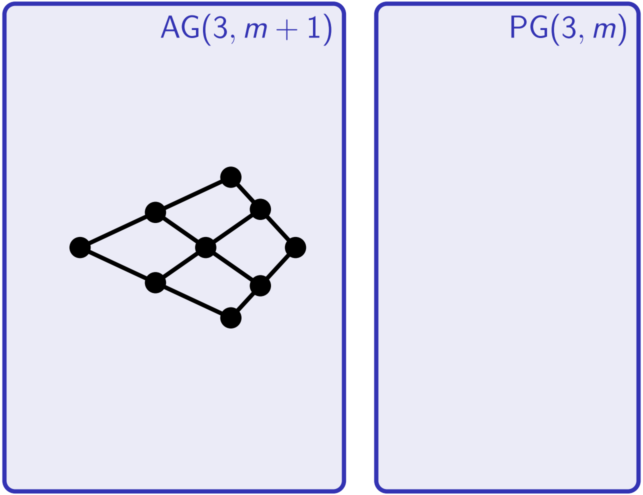

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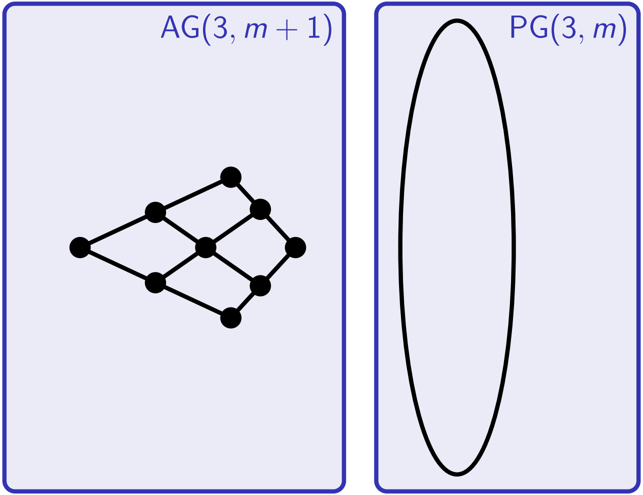

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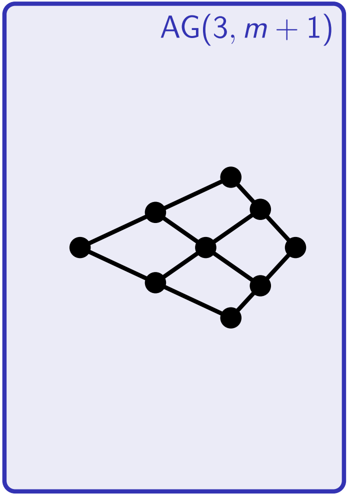

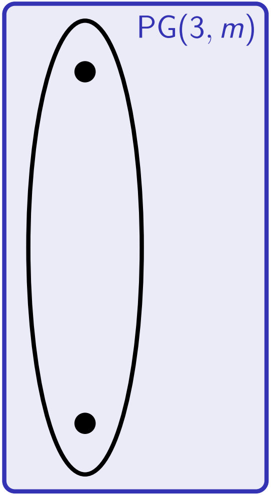

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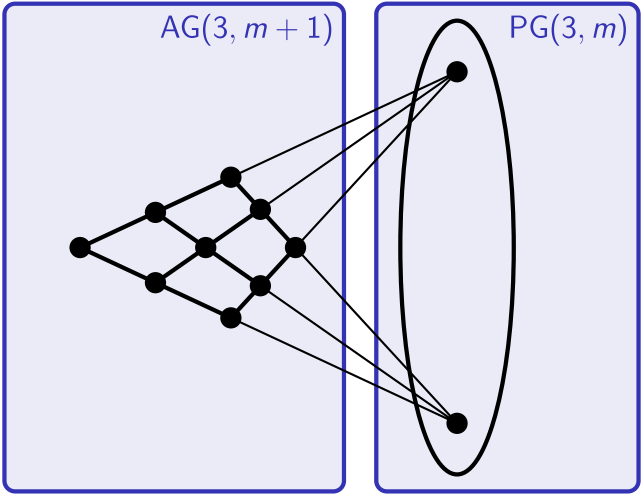

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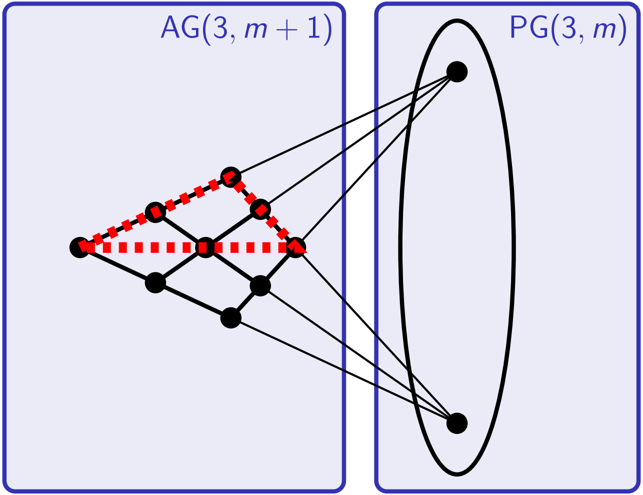

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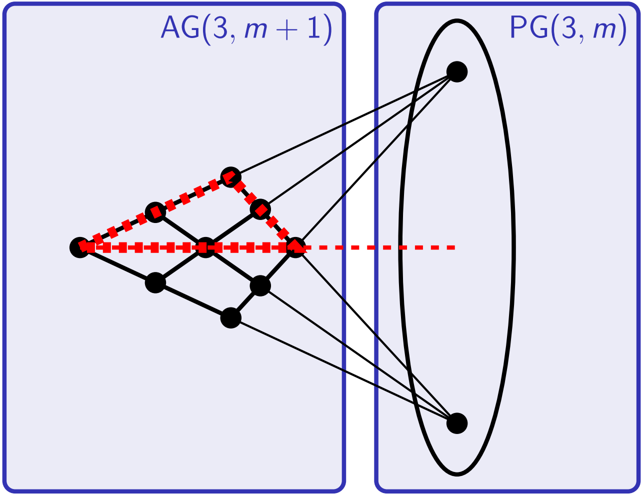

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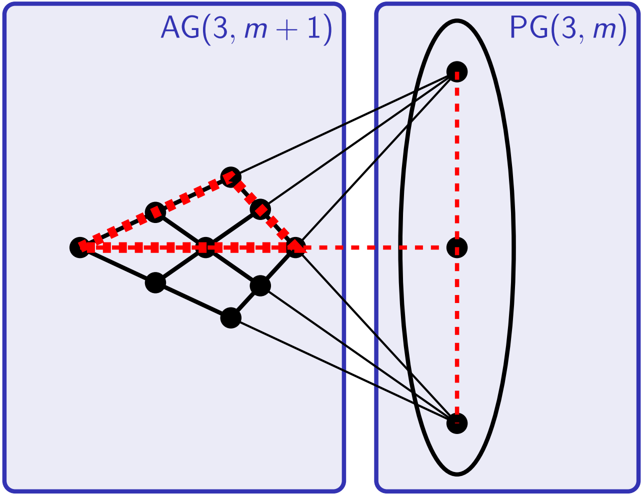
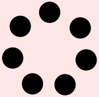


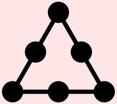
Open problems

Conjecture (Mayhew, Newman, Welsh, Whittle (2011))

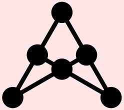
N sparse paving ⇒ almost every matroid has an N-minor.



Prove conjecture for $U_{3,7}$, and in general for $U_{r,n}$.



Prove conjecture for W^3.



Prove conjecture for $M(K_4)$.