On Inequivalent Representations

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Based on joint work with
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A talk in 4 parts

- I. Matroids, representations
- II. Kahn's Conjecture
- III. The prime field case
- IIII. Arbitrary fields

Part I
Matroids, representations

Linear codes

Definition.
Let $\mathbb{F}$ be finite field. $C \subseteq \mathbb{F}^n$ is an $[n, k, d]$ *linear code* over $\mathbb{F}$ if

(i) C is linear subspace of dimension k

(ii) The minimum weight is $\geq d$

Definition.
For $c \in \mathbb{F}^n$, *support* of c:

$$\|c\| := \{i \in [n] : c_i \neq 0\}.$$

The *weight* of c: size of support.

Linear codes

Definition.
Let $\mathbb{F}$ be finite field. $C \subseteq \mathbb{F}^E$ is an $[E, k, d]$ *linear code* over $\mathbb{F}$ if

(i) C is linear subspace of dimension k

(ii) The minimum weight is $\geq d$

Definition.
For $c \in \mathbb{F}^E$, *support* of c:

$$\|c\| := \{i \in E : c_i \neq 0\}.$$

The *weight* of c: size of support.

Matroids

Definition: *Elementary word:* $c \neq \underline{0}$, inclusionwise minimal support.

Theorem.
Define

$$\mathcal{C}^* := \{\|c\| : c \in C, \textit{elementary}\}.$$

Then $\mathcal{C}^*$ is set of cocircuits of a matroid, $M(C)$.

Matroids

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Define

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Then $\mathcal{C}^*$ is set of cocircuits of a matroid, $M(C)$.

(Co)circuit axioms

$\mathcal{C}^*$ is set of cocircuits of a matroid if and only if

- $\emptyset \notin \mathcal{C}^*$
- $C, D \in \mathcal{C}^*$ and $C \subseteq D$ then $C = D$
- $C, D \in \mathcal{C}^*$, $C \neq D$, $e \in C \cap D$, then $(C \cup D) - e$ contains a cocircuit

Usual suspects

Duality/dual code $C^\perp := \{d : \langle c, d\rangle = 0 \ \forall c \in C\}$

$$M(C^\perp) = M(C)^*$$

Deletion/puncturing $C \setminus e$: remove coordinate indexed by e from each word

$$M(C\setminus e) = M(C)\setminus e$$

Contraction/shortening C/e: restrict to words having $c_e = 0$, then remove coordinate

$$M(C/e) = M(C)/e$$

Matrices

Definition.

Generator matrix A: rows form basis of C
Parity check matrix H: rows form basis of $C^{\perp}$

$$c \in C \iff Hc^T = 0$$

Write $M[A]$ for $M(C)$.

The main question:

When is $M(C_1) = M(C_2)$?
When is $M[A_1] = M[A_2]$?

Interpretations:

- Algorithm
- Counting

Algorithm

Theorem (Král' (2007), basically)
If $M[A_1]$ has bounded branch width, can test in polynomial time if $M[A_1] = M[A_2]$.

Conjecture (Geelen, Gerards, Whittle)
There is always a polynomial-time algorithm to test if $M[A_1] = M[A_2]$.

Counting

- Exact, for given matroid: basically previous slide.
- Qualitatively: upper bounds.

Part II
Kahn's Conjecture

Easy cases

Theorem.
If C_1 and C_2 binary, then

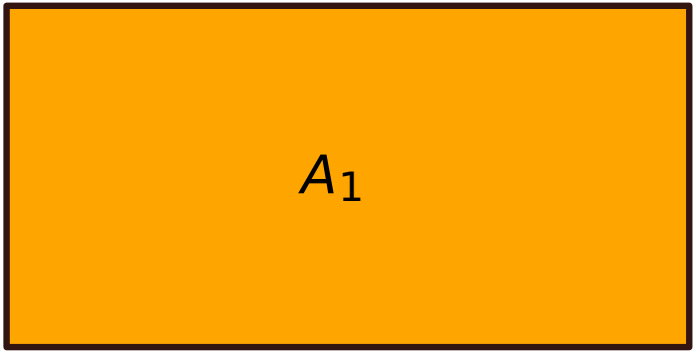
$$M(C_1) = M(C_2) \iff C_1 = C_2.$$

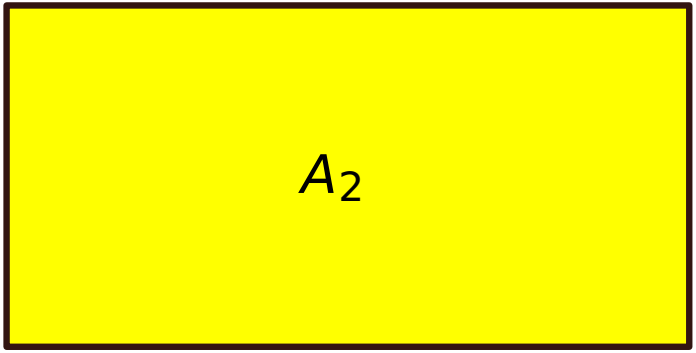
Theorem (Brylawski, Lucas).
If C_1 and C_2 ternary, then

$$M(C_1) = M(C_2) \iff C_1 = C_2'$$

where some coordinates of C_2 were scaled by -1 to get C_2'.

Operations





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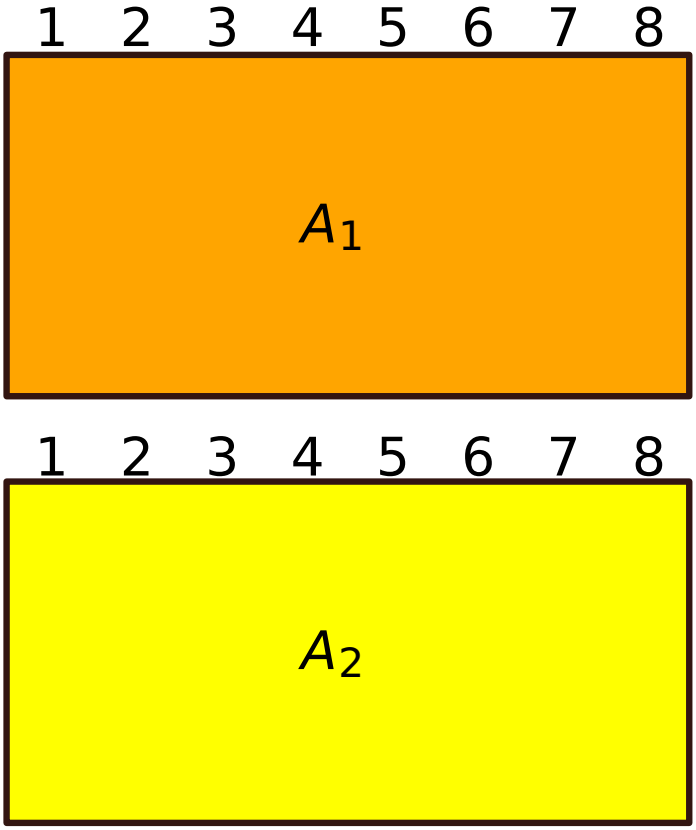

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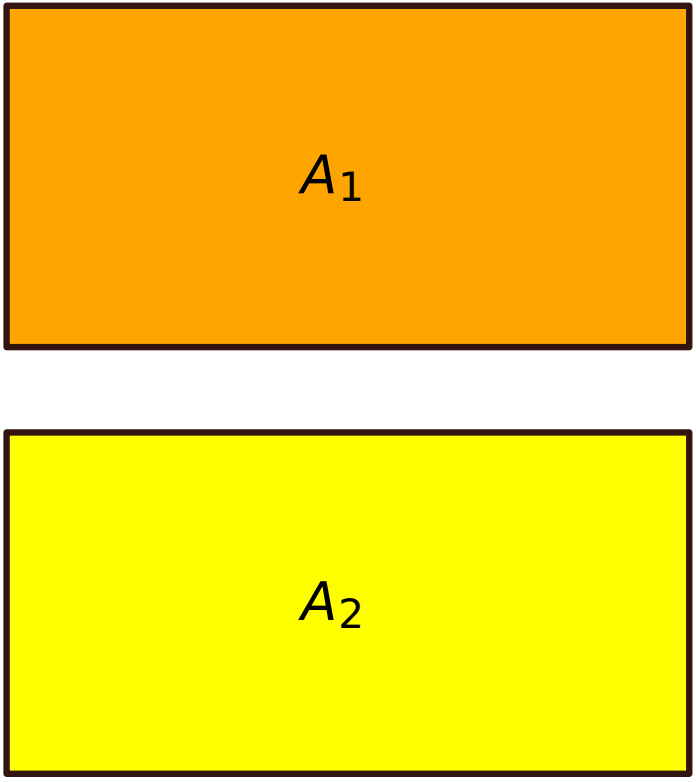

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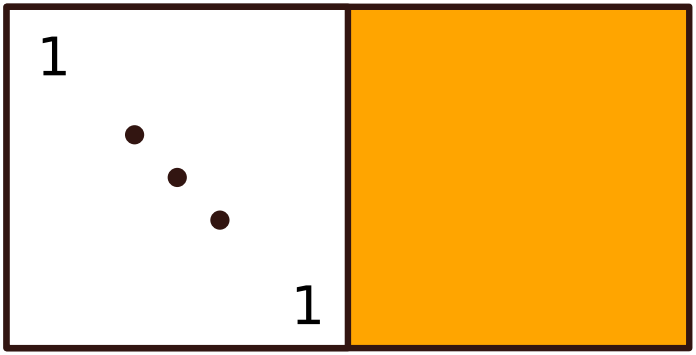

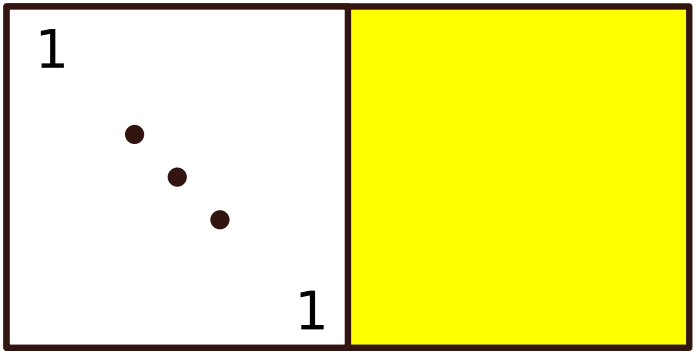

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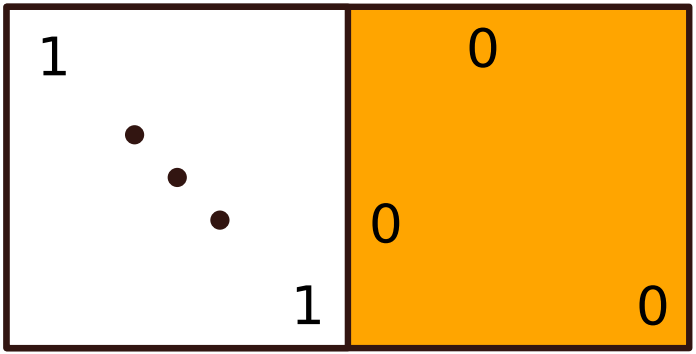

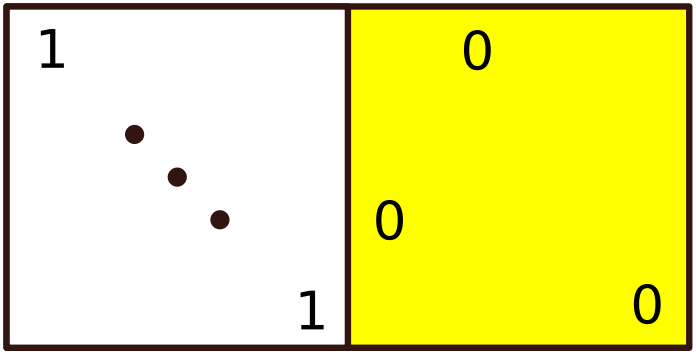

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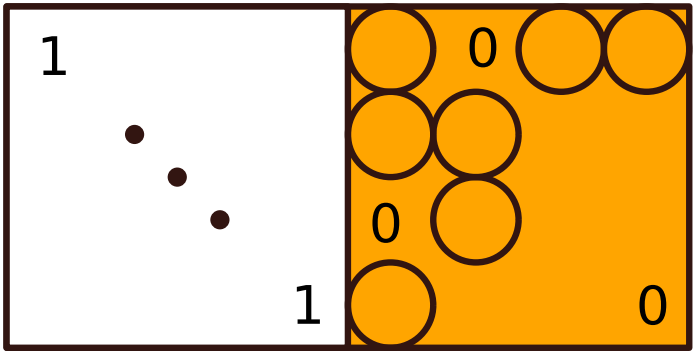

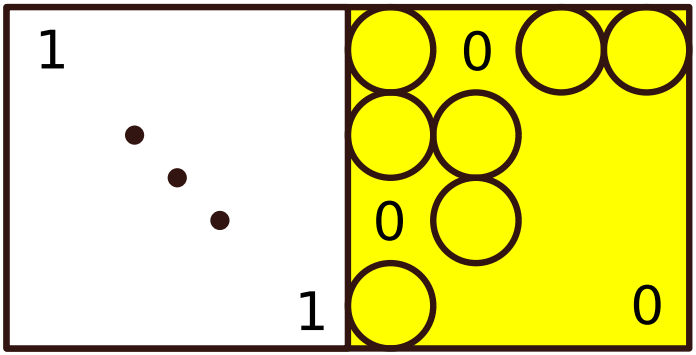

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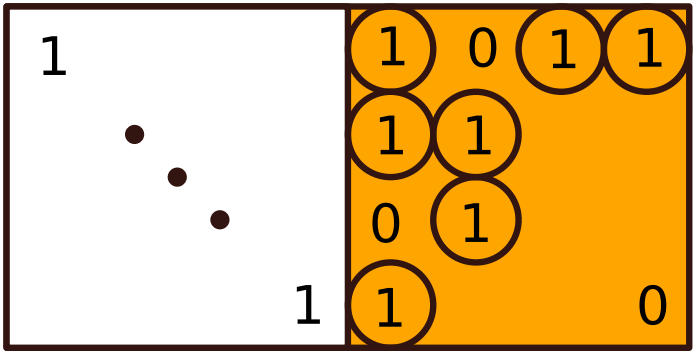
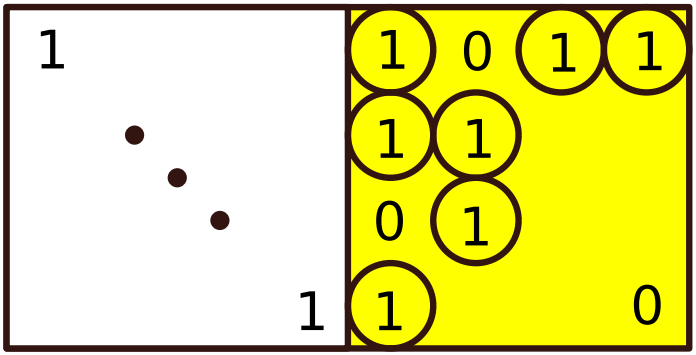




No bound?

$$\begin{bmatrix} 1 & 1 & & & & & \\ 1 & 2 & & & & & \\ & & 1 & 1 & & & \\ & & 1 & 3 & & & \\ & & & & 1 & 1 & \\ & & & & 1 & 2 & \\ & & & & & & \ddots \\ & & & & & & & \ddots \\ & & & & & & & & \ddots \end{bmatrix}$$

Relief

Theorem (Kahn 1981).
If A_1, A_2 quaternary, $M[A_1]$ **3-connected**, $M[A_1] = M[A_2]$ then A_1, A_2 related through

- Row operations
- Column scaling
- Field automorphism

Kahn's Conjecture

Conjecture.
For each $\mathbb{F}$, there is c such that each 3-connected M has at most c inequivalent representations.

Hurray!

Theorem (Oxley, Vertigan, Whittle 1995).
For GF(5), each 3-connected M has at most 6 inequivalent representations.

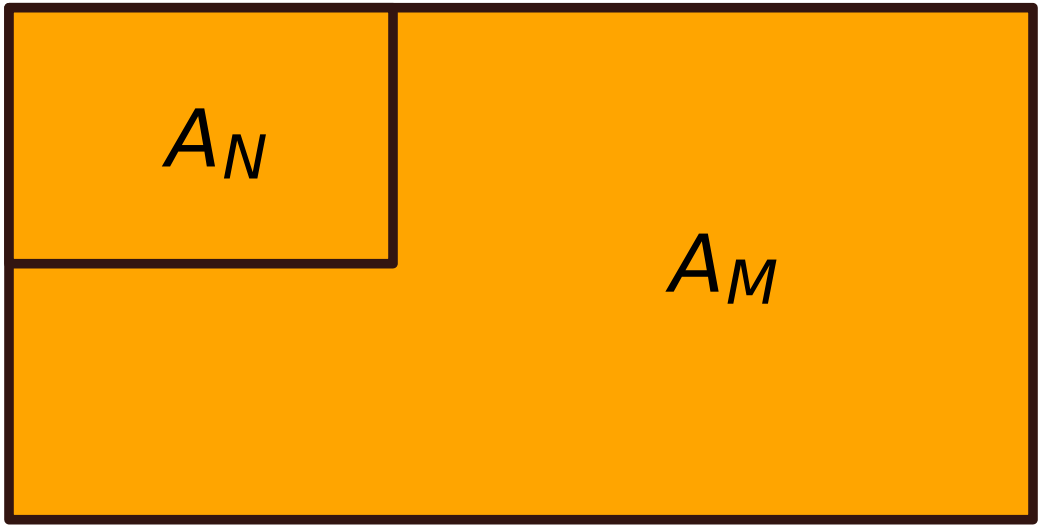
Alas.

Theorem (Oxley, Vertigan, Whittle 1995).
For GF(5), each 3-connected M has at most 6 inequivalent representations.

Theorem (Oxley, Vertigan, Whittle 1995).
Kahn's Conjecture is *false* for all larger fields.

Stabilizers

Stabilizers



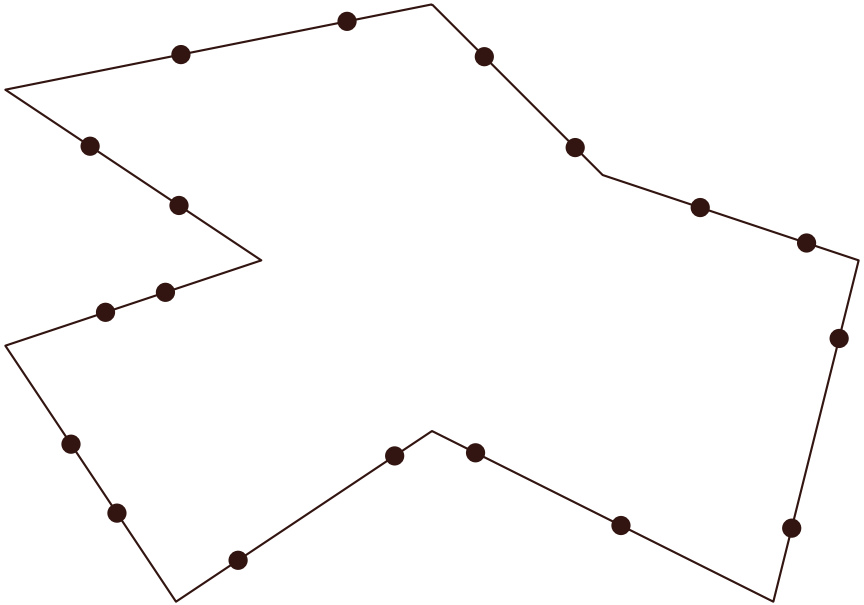
Stabilizer

Whittle's Stabilizer Theorem (1999).
If N is not a stabilizer for a class, it will show after 2 steps.

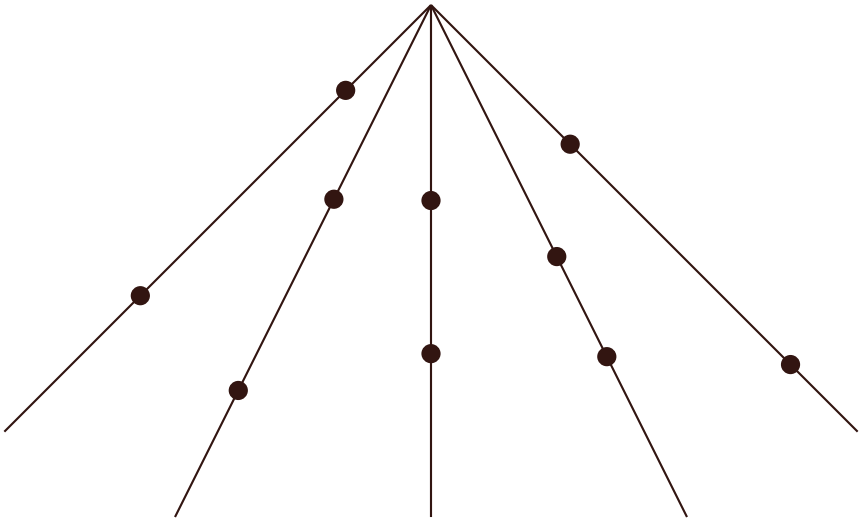
Corollary 4.
$U_{2,4}$ stabilizes 3-connected quaternary matroids.

Corollary 5.
$U_{2,5}$, $U_{3,5}$ stabilize 3-connected quinary matroids. $U_{2,4}$ stabilizes remaining 3-connected quinary matroids.

Counterexamples: swirls



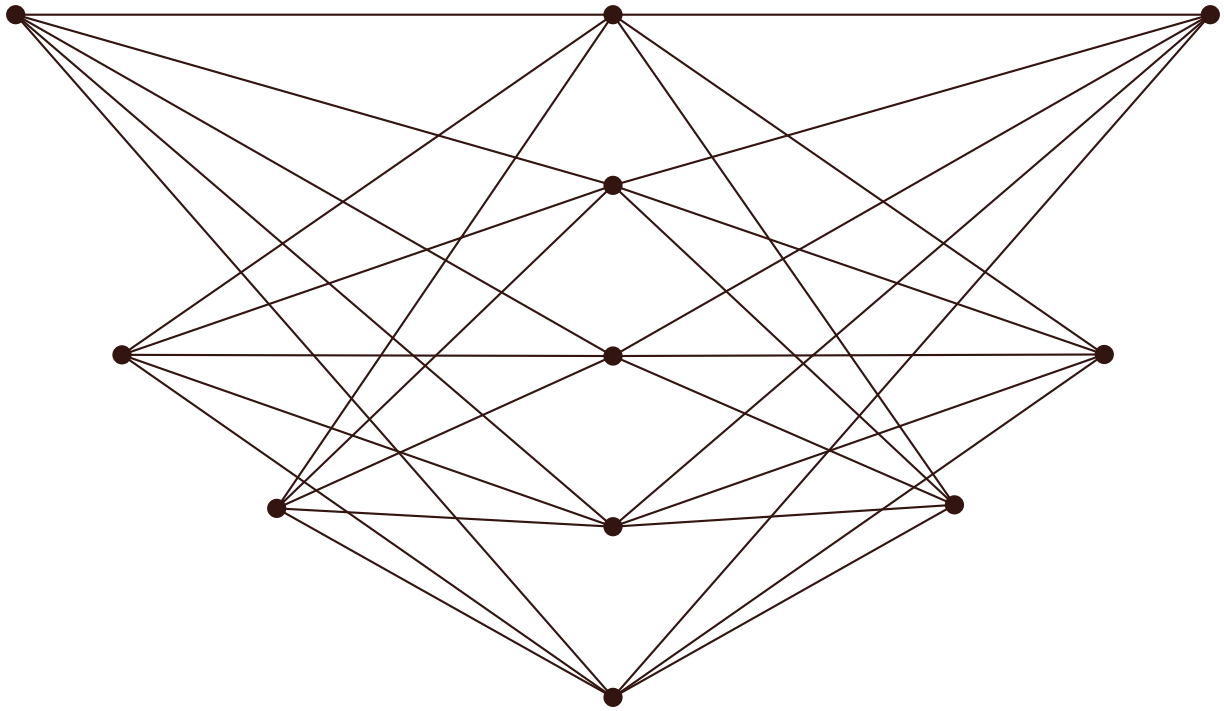
Counterexamples: spikes



New conjecture

Conjecture.
For each $\mathbb{F}$, there is c such that each 4-connected M has at most c inequivalent representations.

Counterexamples: maces



New conjecture

Conjecture.
For each $\mathbb{F}$, there are c, k such that each k-connected M has at most c inequivalent representations.

Part III
The prime field case

The result

Theorem (Geelen, Whittle 2011+).
For each GF(p) there is c such that each 4-connected M has at most c inequivalent representations.

Proof (idea).
Characterize "worst offenders," bound their size. Details: about 275 pages.

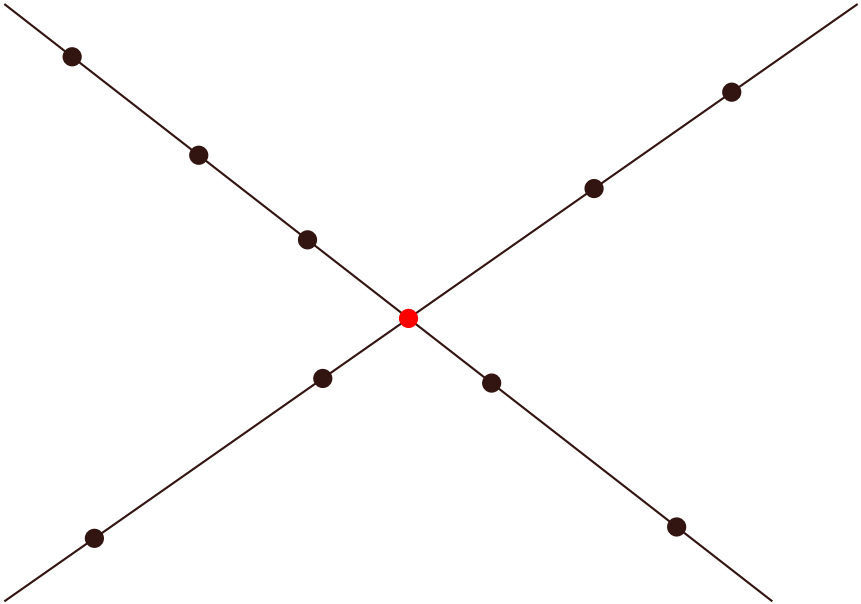
Freedom

Definition.
$\{e, f\}$ is *clonal pair* if exchanging them is isomorphism.

Definition.
e is *fixed* if no extension has independent clonal pair $\{e, f\}$.

Corollary.
If e fixed, then representation of $M \backslash e$ extends uniquely.

Freedom



k-coherence

Definition.
M is *k-coherent* if 3-connected and has no swirl-like flower with $\geq k$ petals.

Worst offenders

Definition.

A *k-skeleton* for GF(p) is 3-connected M with

- M is not wheel, whirl of rank ≥ 3
- If e fixed then $M \backslash e$ not k-coherent
- If e cofixed then M/e not k-coherent

Chain theorem for skeletons

Theorem (Geelen, Whittle 2011+).
Let M be nonempty k-skeleton. One of these situations applies:

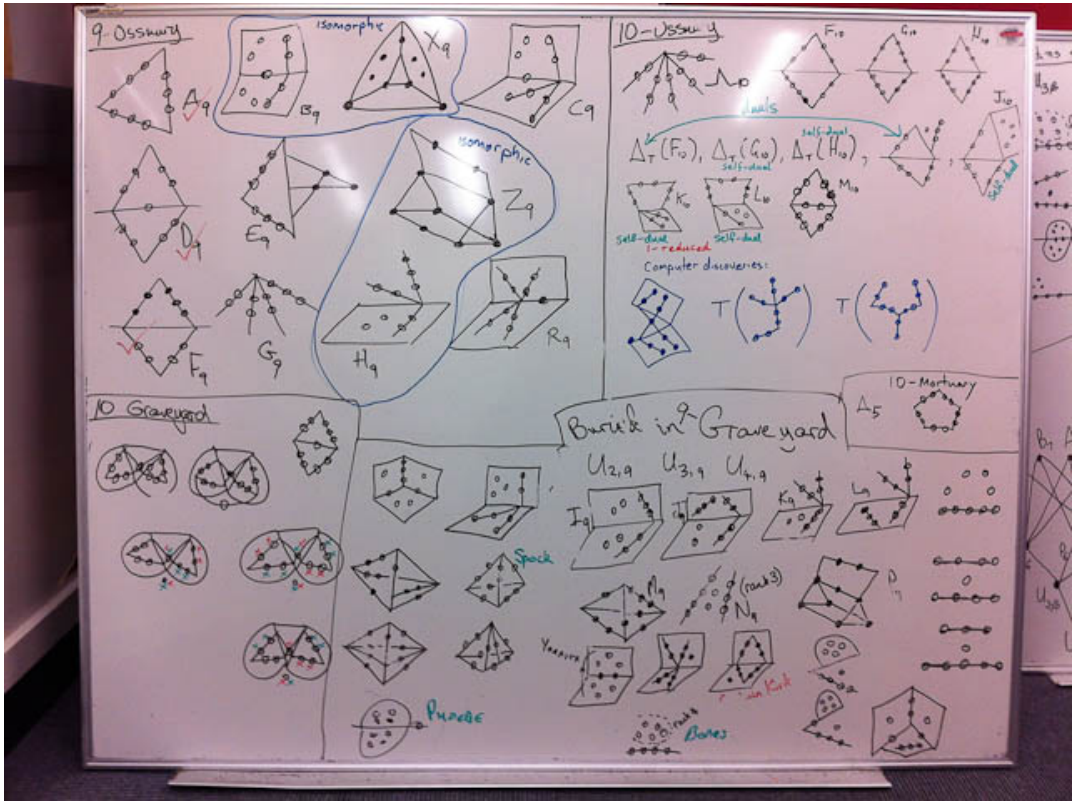
(i) $M \backslash e$ or M/e is k-skeleton for some e

(ii) $M/p \backslash q$ is k-skeleton for some clonal pair $\{p, q\}$

(iii) Special structure stuff that needs $\geq$ 20ish elements and 3- or 4-element reductions.

Work in progress (Hall, Mayhew, vZ).
Use this (and a computer) to get *explicit* bound on 5-coherent, GF(7).

How to do it by hand?



How to do it in Sage?

```
def is_clonal_pair(M,e,f):
    morphism = {}
    for x in M.groundset():
        morphism[x] = x
    morphism[e] = f
    morphism[f] = e
    return M.is_isomorphism(M, morphism)
```

Size	4	5	6	7	8	9	10	11	12	13
Skeletons	1	2	4	10	28	18	20	16*	28*	???

Part IV
Arbitrary fields

New conjecture

Conjecture.
For each finite $\mathbb{F}$, there are c, k such that each k-connected M has at most c inequivalent representations.

"Theorem" (Geelen, Gerards, Huynh, vZ).
True!

(Note: proof gives horrible bounds on c, k).

New connectivity

Definition.
M is (f, k)-connected if, for each l-separation (A, B), $l \leq k$,

$$\min\{|A|, |B|\} \leq f(l).$$

Lemma.
If M is (f, k)-connected, and f non-decreasing, then $M \backslash e$ or M/e is $(2f, k)$-connected.

Lemma.
If M is simple, (f, k)-connected, and critical, then $|E(M)| \leq 2^{r(M)}$.

Chain theorem

“Theorem” (Geelen, Gerards, Huynh, vZ).
Let f grow fast, M huge, (f,k)-connected. Then have (f,k)-connected M' with

$$|E(M)| - |E(M')| \leq 2.$$

Moreover, can protect a small set X.

Constraining freedom

'Theorem' (Geelen, vZ).
If $\mathcal{T}$ is tangle of M (representable over GF(q)), then the nonfixed elements are in a $\mathcal{T}$-small set of bounded order.

Observation.
(f, k)-connectivity gives natural tangle!

Slides at
http://www.math.princeton.edu/~svanzwam/

The End